Applied Geometry: Foldings and Unfoldings

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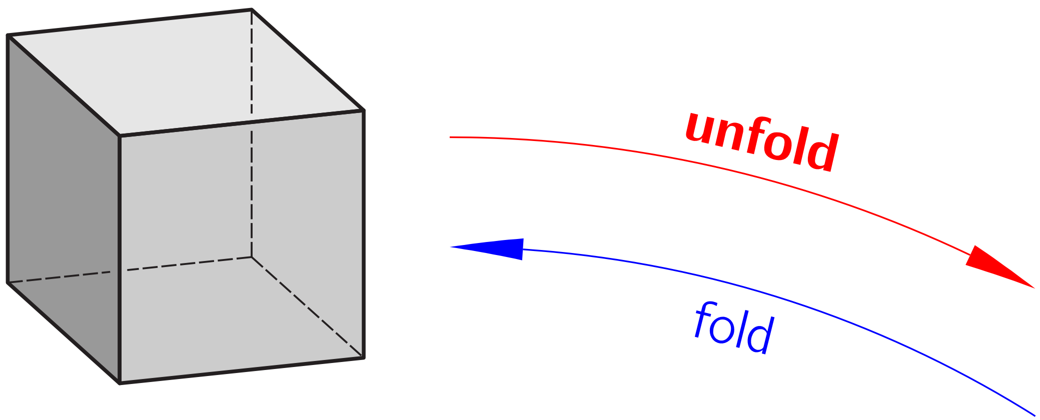
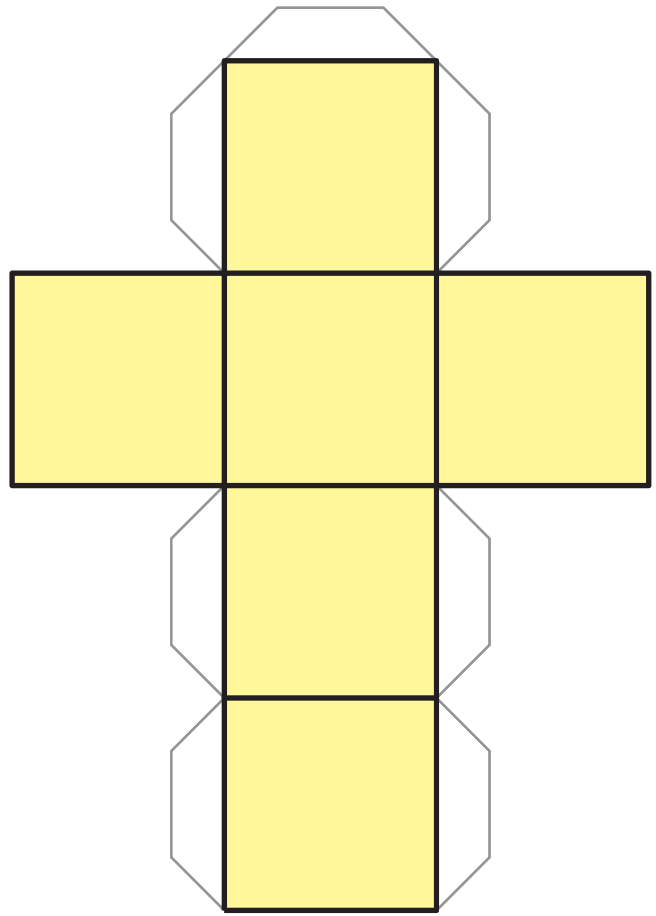
1. Unfolding and folding

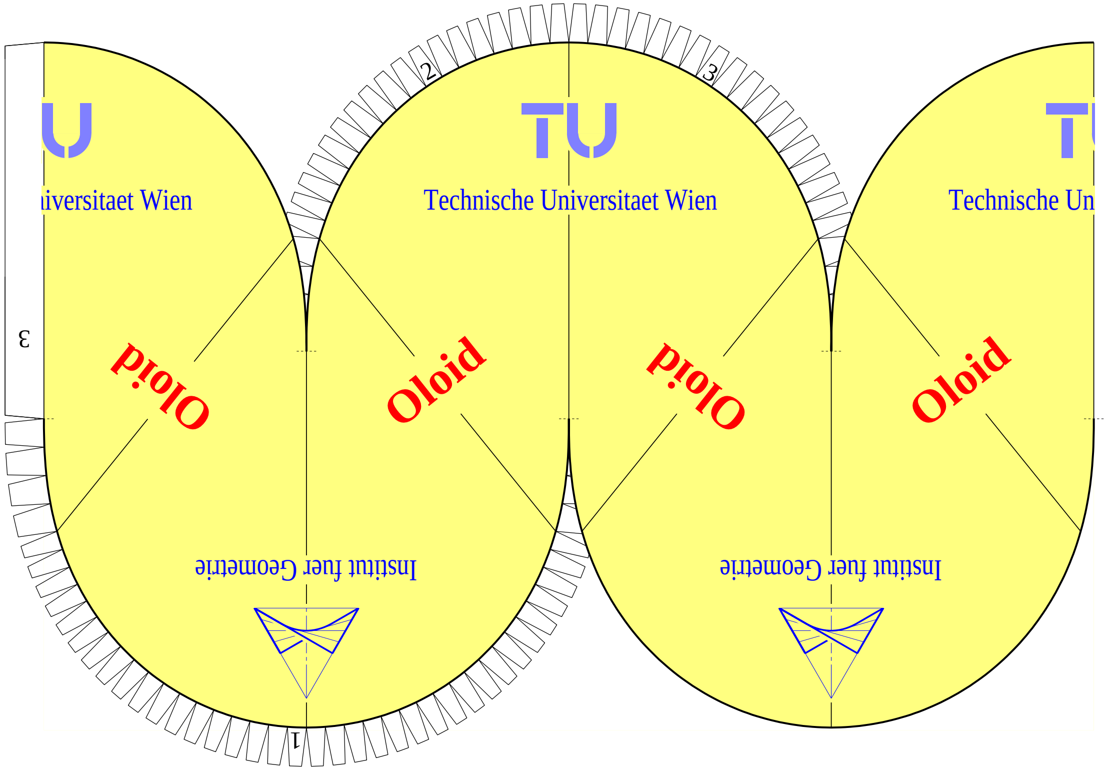


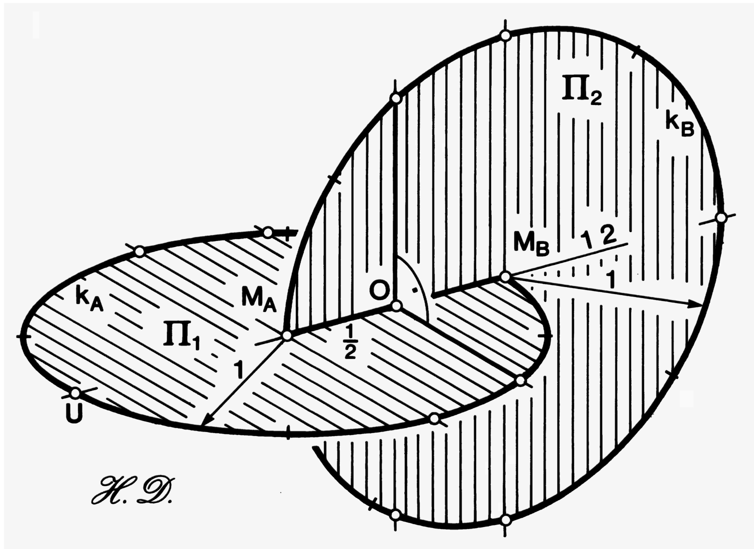


There are standard procedures provided for the construction of the **unfolding** (development, net) of polyhedra or developable surfaces.

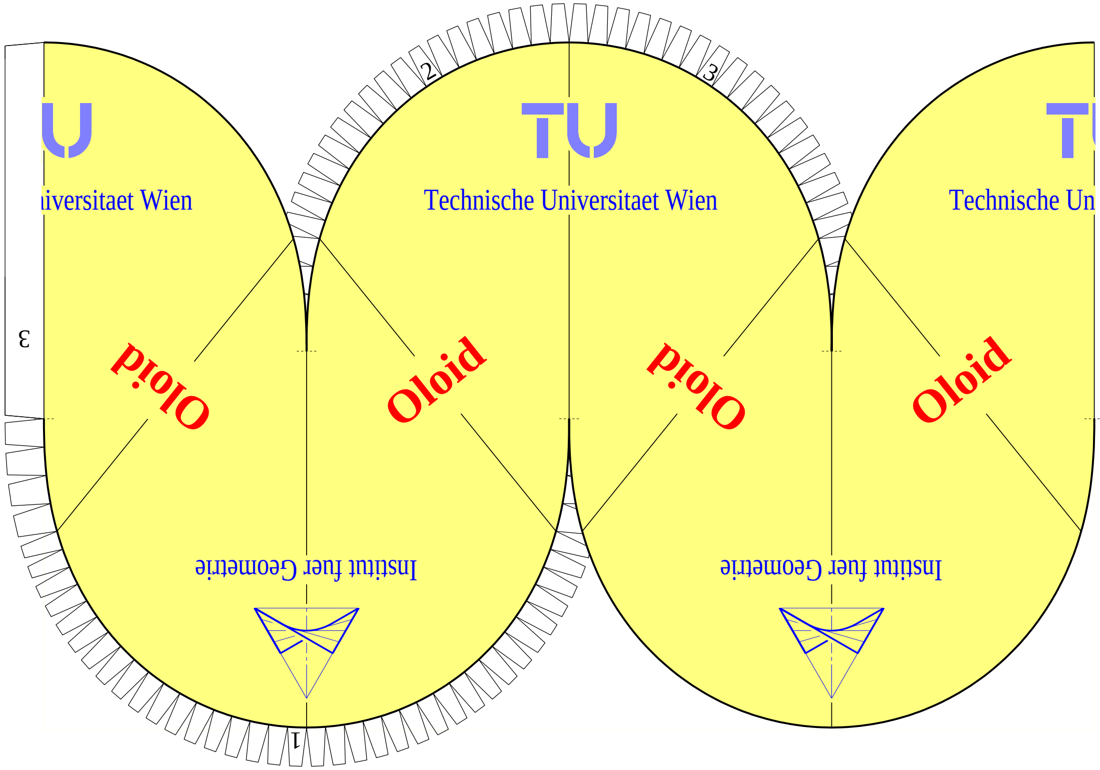
The result is unique, apart from the placement of the different components, and it shows the intrinsic metric of the spatial structure.

E.g., the unfolding of the **Oloid**





Developable surfaces are ruled surfaces with vanishing Gaussian curvature



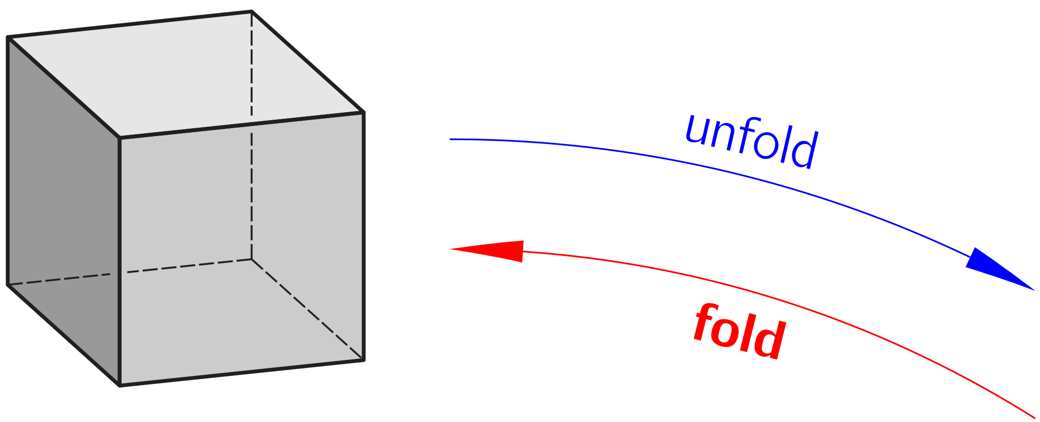
Oloid:

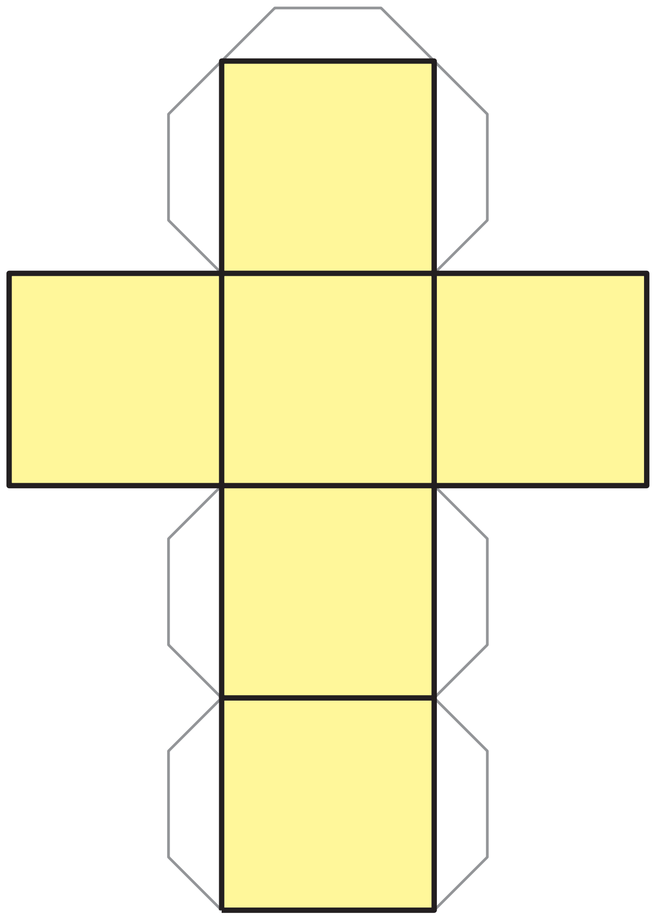
arc-length parametrization of the unfolding of the circles:

$$x(s) = \frac{2\sqrt{3}}{9}\left[\arccos\frac{\sqrt{2}\cos s}{\sqrt{1+\cos s}} - \frac{\sqrt{2(1-\cos s)(1+2\cos s)}}{(1+\cos s)}\right]$$

$$y(s) = \frac{\sqrt{3}}{9}\left[\ln\frac{2}{1+\cos s} + \frac{11+7\cos s}{1+\cos s}\right].$$

1. Unfolding and folding





The inverse problem, i.e., the determination of a folded structure from a given unfolding is more complex. In the smooth case we obtain a continuum of bent poses.

In the polyhedral case the computation leads to a system of algebraic equations. Also here the corresponding spatial object needs not be unique.

1. Unfolding and folding

Only if the polyhedron bounds a **convex** solid then the result is unique, due to Aleksandr Danilovich **Alexandrov** (1941).

In this case, for each vertex the sum of intrinsic angles for all adjacent surfaces is $< 360^\circ$ (= convex intrinsic metric).

Theorem: [Uniqueness Theorem]
For any convex intrinsic metric there is a unique convex polyhedron.

1. Unfolding and folding

If convexity is not required the unfolding of a polyhedron needs not define its spatial shape uniquely !

Definition 1: A polyhedron is called **globally rigid** if its intrinsic metric defines its spatial form uniquely — up to movements in space.

1. Unfolding and folding

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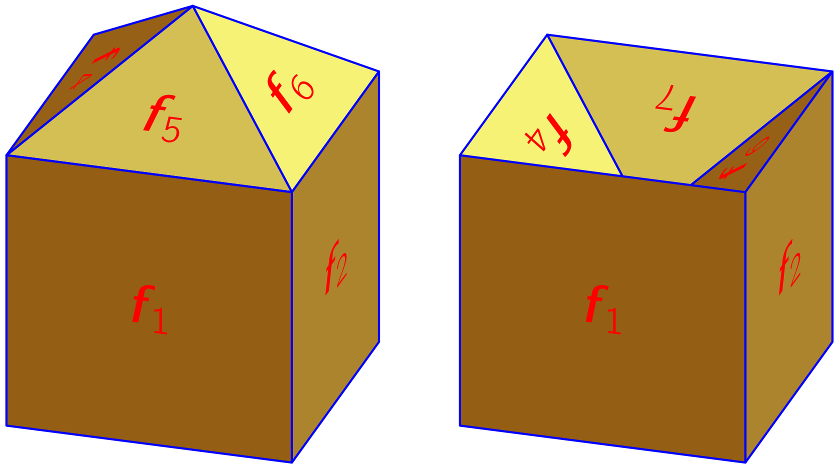
Definition 1: A polyhedron is called **globally rigid** if its intrinsic metric defines its spatial form uniquely — up to movements in space.

e.g., a tetrahedron



A **flipping** (or snapping) polyhedron admits two sufficiently close realizations – by applying a slight force.

1. Unfolding and folding

Definition 2: A polyhedron is called **(continuously) flexible** if there is a *continuous family* of mutually incongruent polyhedra sharing the intrinsic metric. Each member of this family is called a **flexion**.

1. Unfolding and folding

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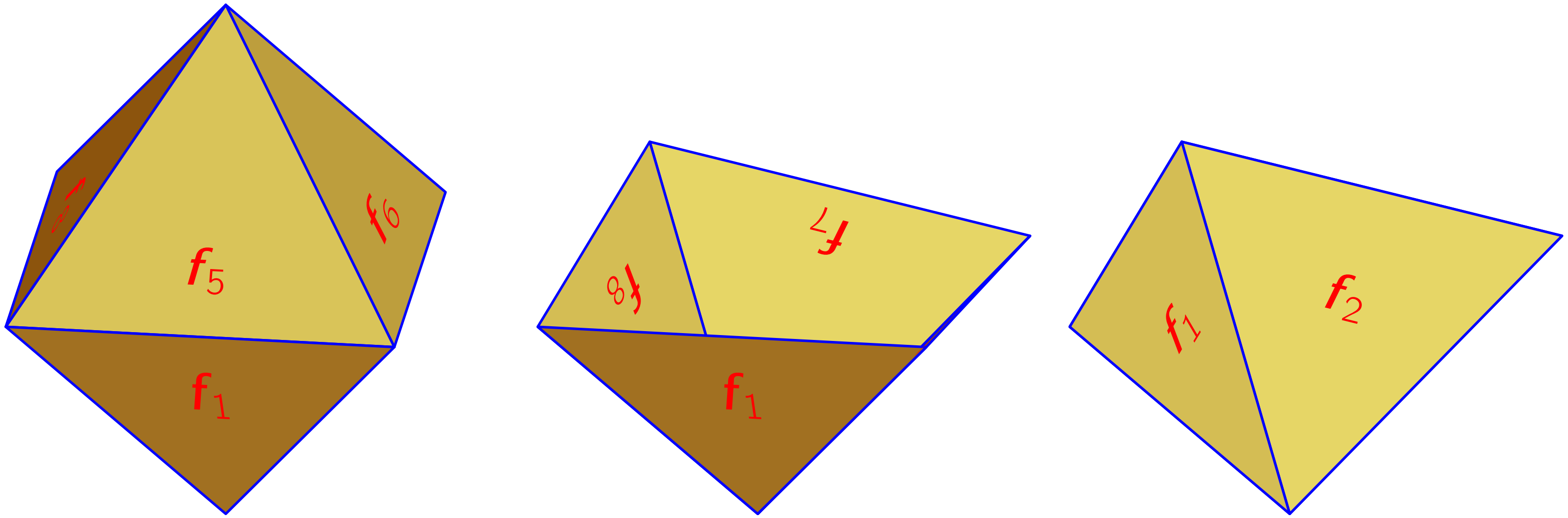
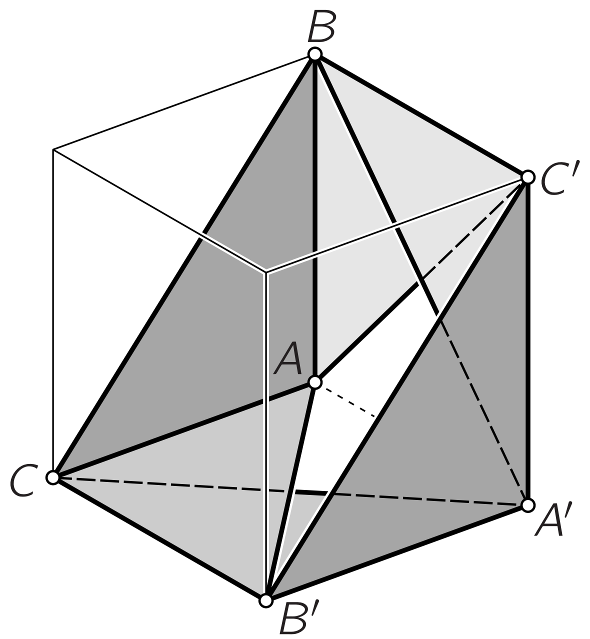
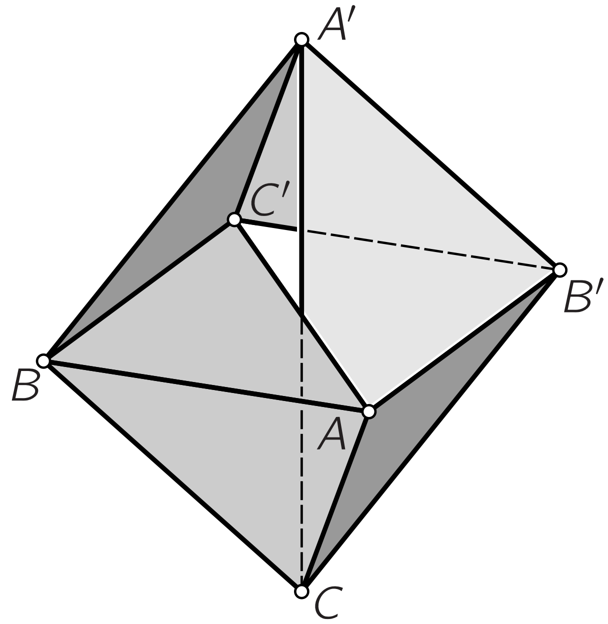


Even a regular octahedron is flexible — after being re-assembled. The regular pose on the left hand side is called **locally rigid**.

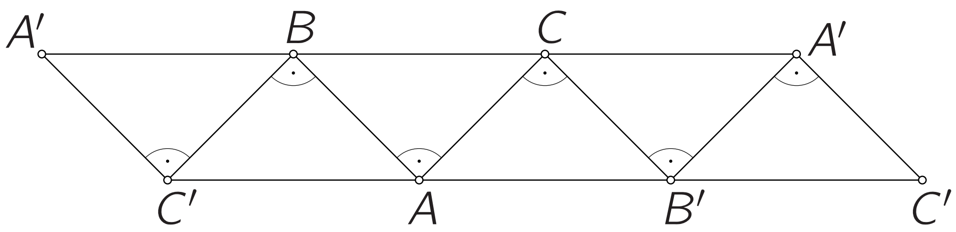
1. Unfolding and folding

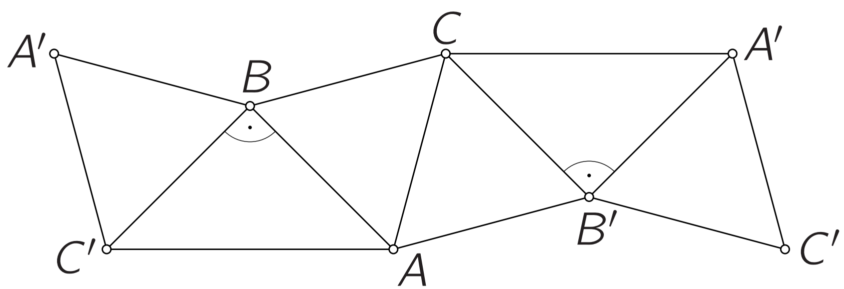




Two particular examples of flexible octahedra where two faces are omitted. Both have an axial symmetry (types 1 and 2)

Below: Nets of the two octahedra.





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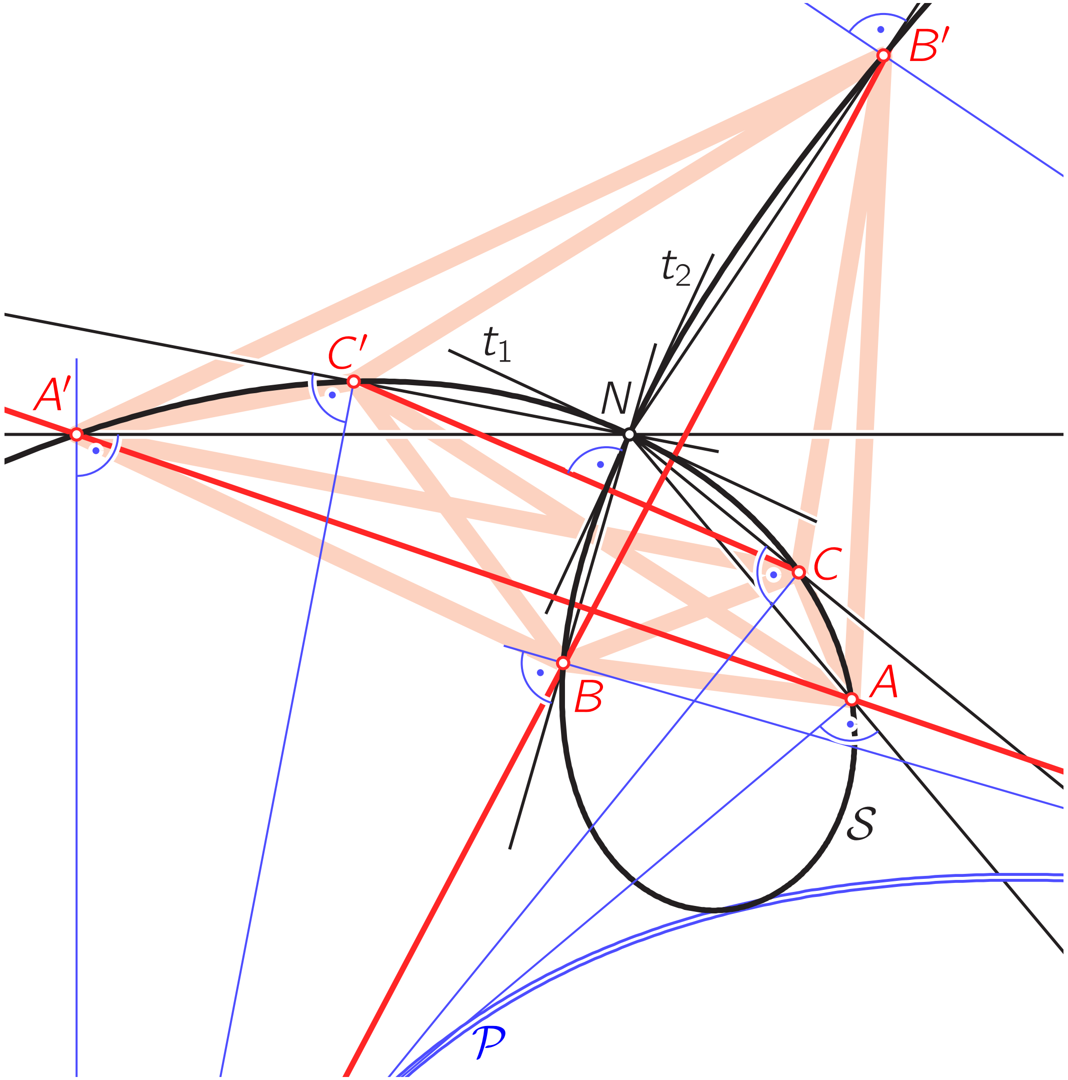
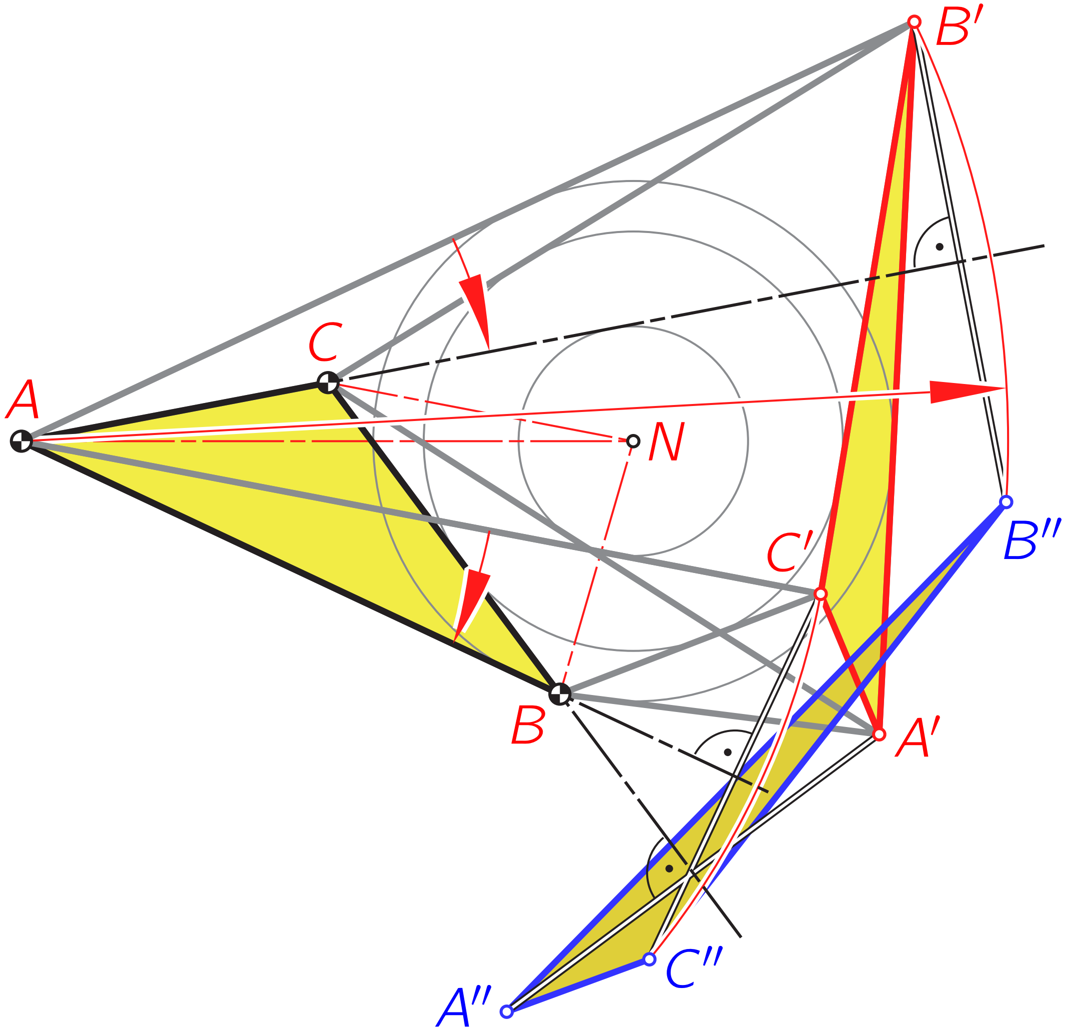


According to R. Bricard (1897) there are three types of flexible octahedra (four-sided double-pyramids).

The first two have axis or plane of symmetry. Those of type 3 admit two flat poses. In each such pose, the pairs (A, A'), (B, B'), and (C, C') of opposite vertices are associated points of a strophoid $\mathcal{S}$.

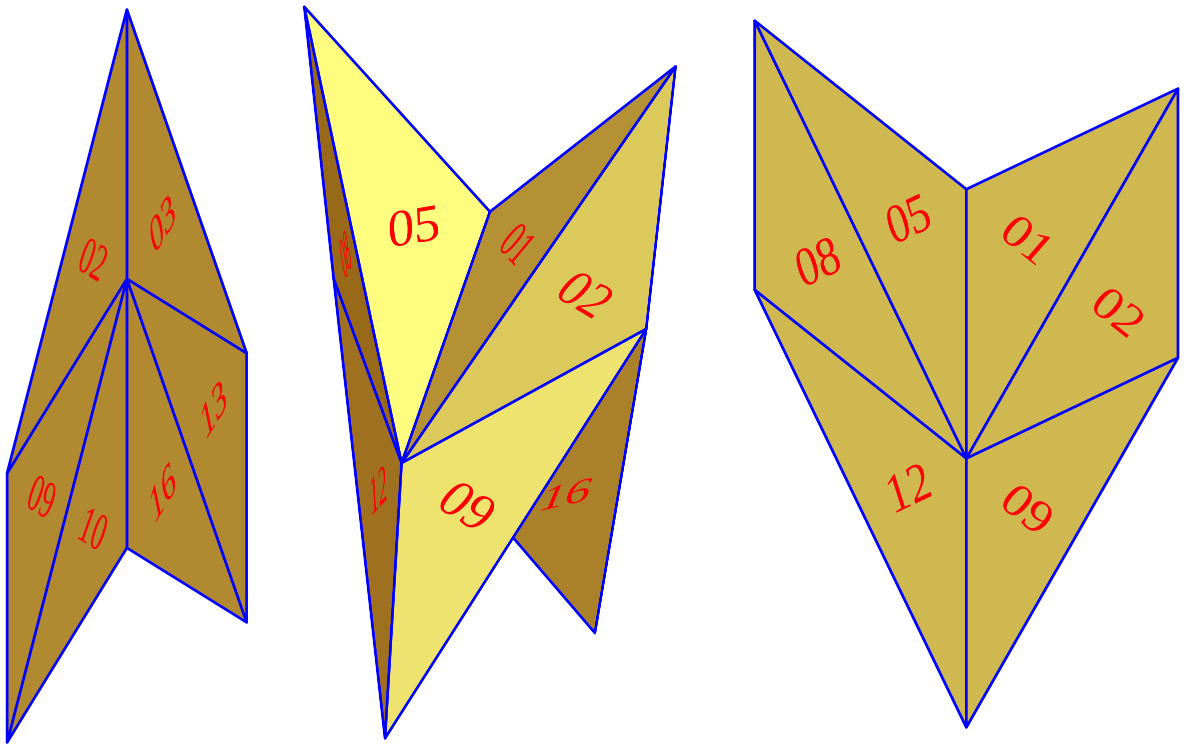
1. Unfolding and folding



According to Bricard's construction, all bisectors must pass through the midpoint N of the concentric circles.

The two flat poses of a type-3 flexible octahedron, when ABC remains fixed.

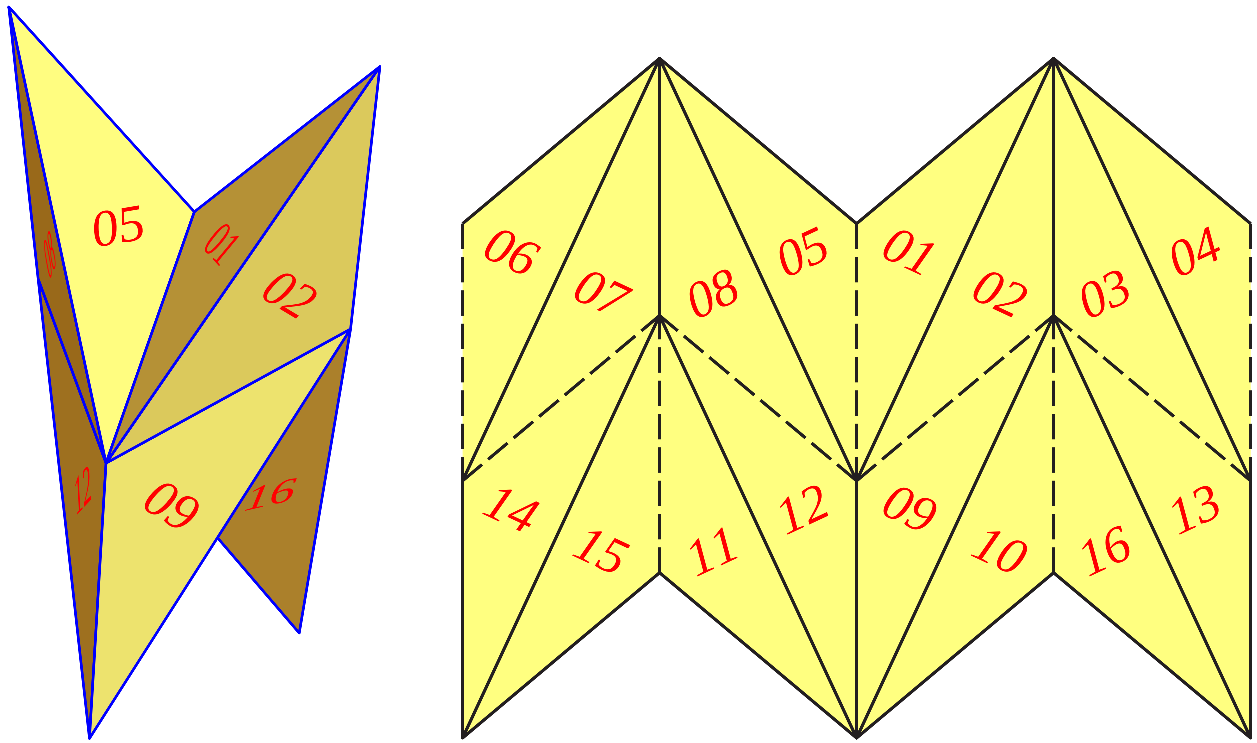
1. Unfolding and folding



This polyhedron called *"Vierhorn"* is locally rigid, but can flip between its spatial shape and two flat realizations in the planes of symmetry (W. Wunderlich, C. Schwabe).

At the science exposition *"Phänomena"* 1984 in Zürich this polyhedron was exposed and falsely stated that this polyhedron is flexible.

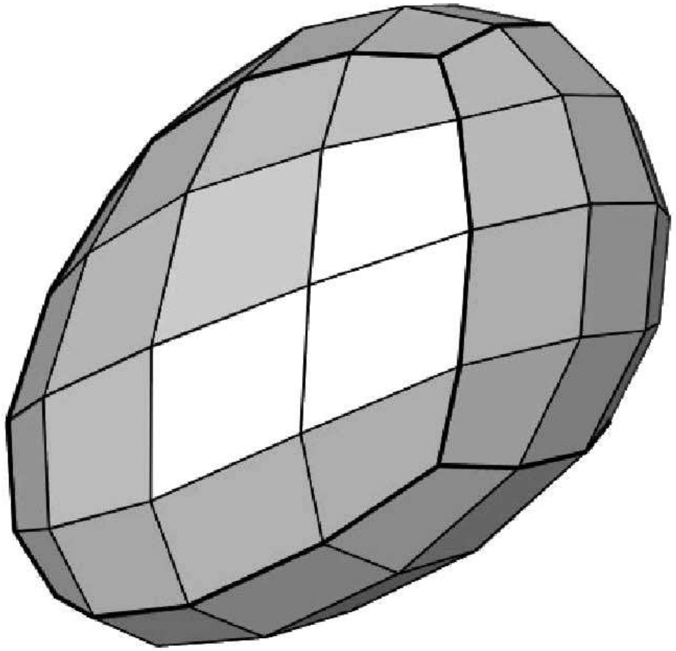
1. Unfolding and folding



the “Vierhorn” and its unfolding

Wolfram MathWorld: A flexible polyhedron which flexes from one totally flat configuration to another, passing through intermediate configurations of positive volume.

2. Flexible quad-meshes



A **polygonal mesh** is a simply connected subset of a polyhedral surface (sphere-like or with boundary) consisting of (not necessarily planar) polygons, edges and vertices in the Euclidean 3-space.

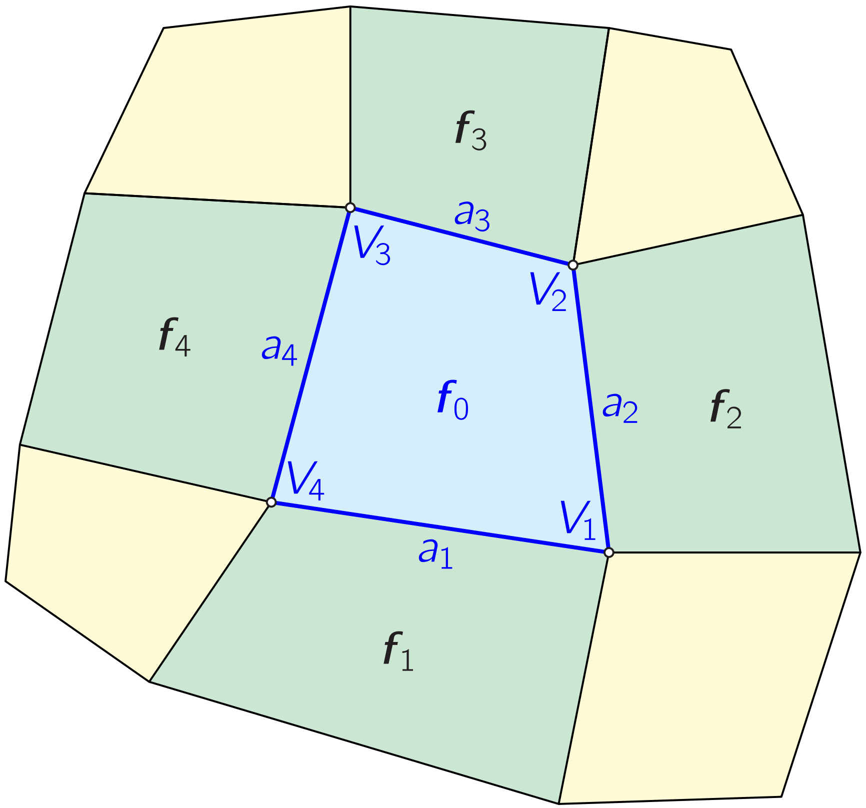
The edges are either *internal* when they are shared by two faces, or they belong to the boundary of the mesh.

When all polygons are quadrangles, then it is called a **quadrilateral surface**.

2. Flexible quad-meshes

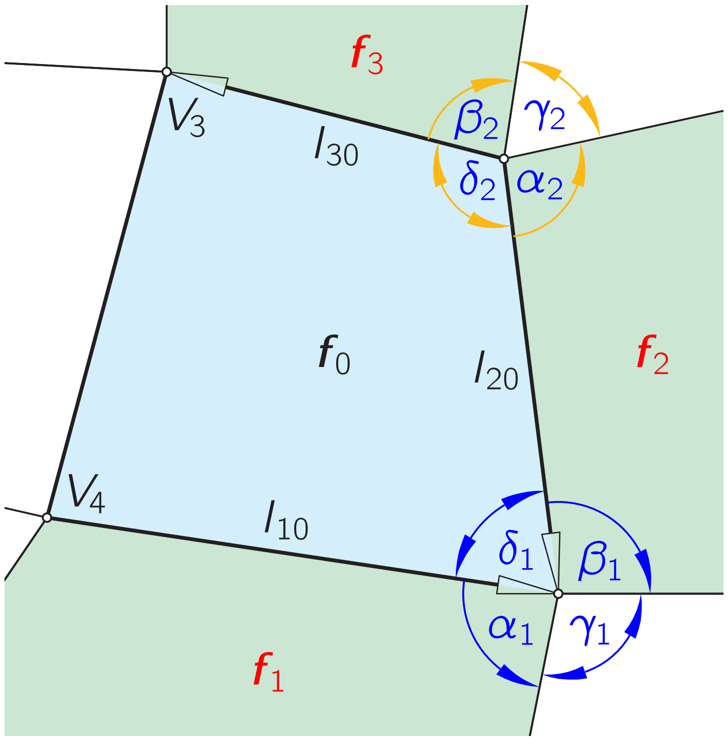
In modern architecture, most freeform surfaces are designed as *polyhedral surfaces* – like the Capital Gate in Abu Dhabi, built by the Austrian company Waagner Biro (160 m high, 18° inclination)

2. Flexible quad-meshes

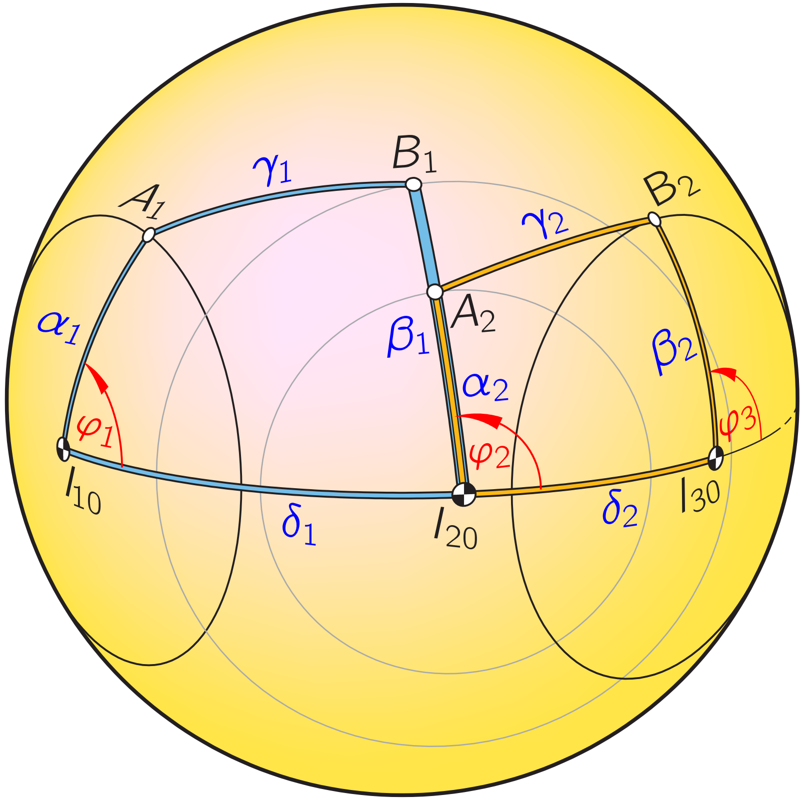


Under which conditions is this 3×3 mesh continuously flexible ?

2. Flexible quad-meshes



Transmission from f_1 to f_3 via the quadrangle f_2



Composition of two spherical four-bar linkages

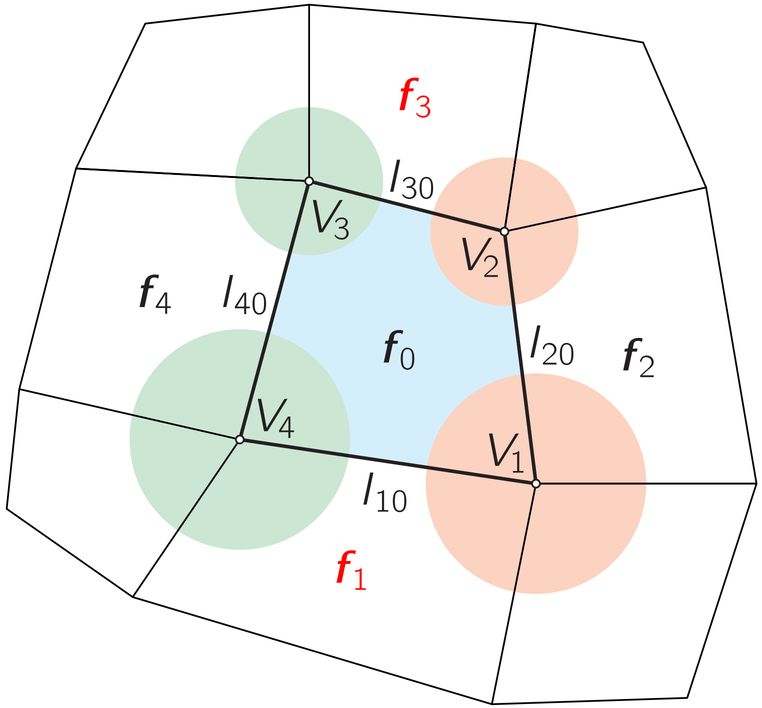
2. Flexible quad-meshes

A Kokotsakis mesh is continuously flexible

$\Longleftrightarrow$

the transmission from $\boldsymbol{f}_1$ to $\boldsymbol{f}_3$ can in two ways be decomposed into two spherical four-bar mechanisms, one via V_1 and V_2, the other via V_4 and V_3.

The internal edges can be arranged in two 'horizontal' (blue) and two 'vertical' edge folds (red).



2. Flexible quad-meshes

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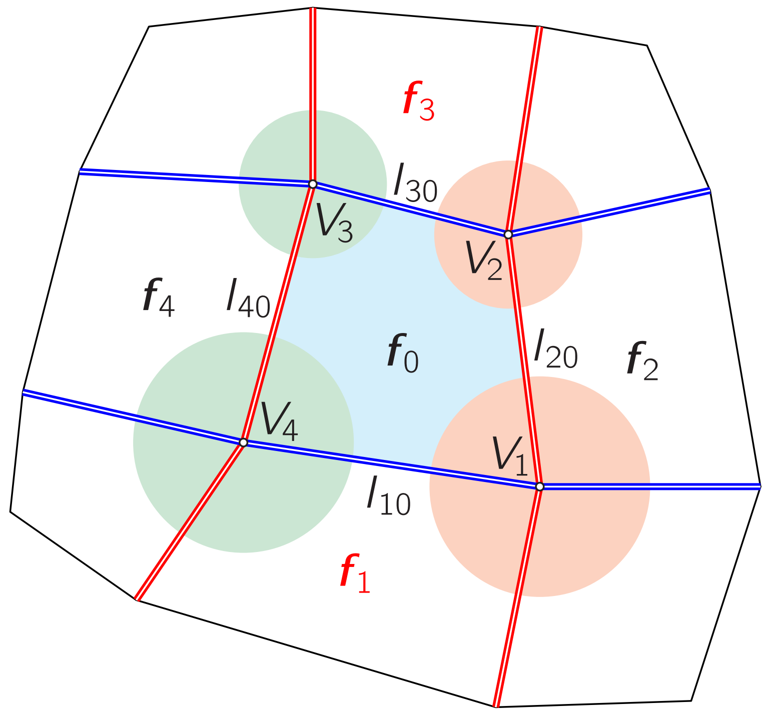

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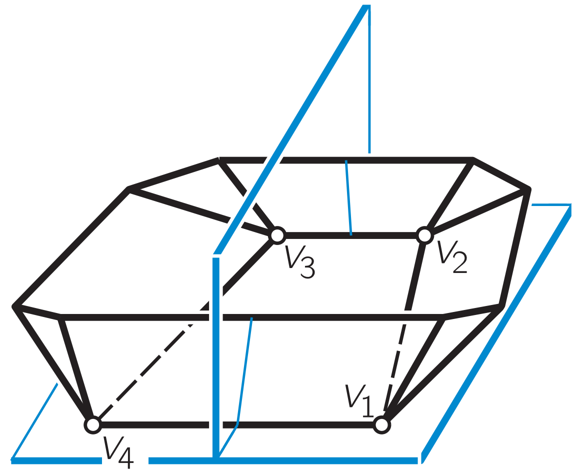
2. Flexible quad-meshes

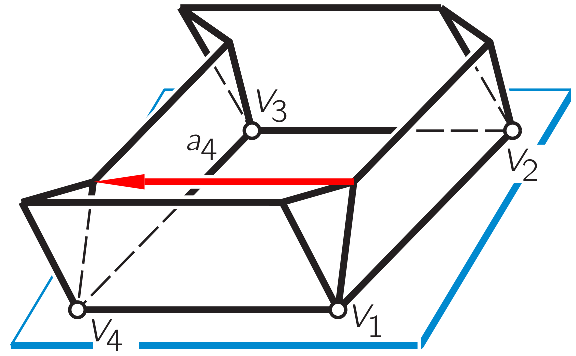
I. Planar-symmetric type (Kokotsakis 1932):

The reflection in the plane of symmetry of V_1 and V_4 maps each horizontal fold onto itself while the two vertical folds are exchanged.

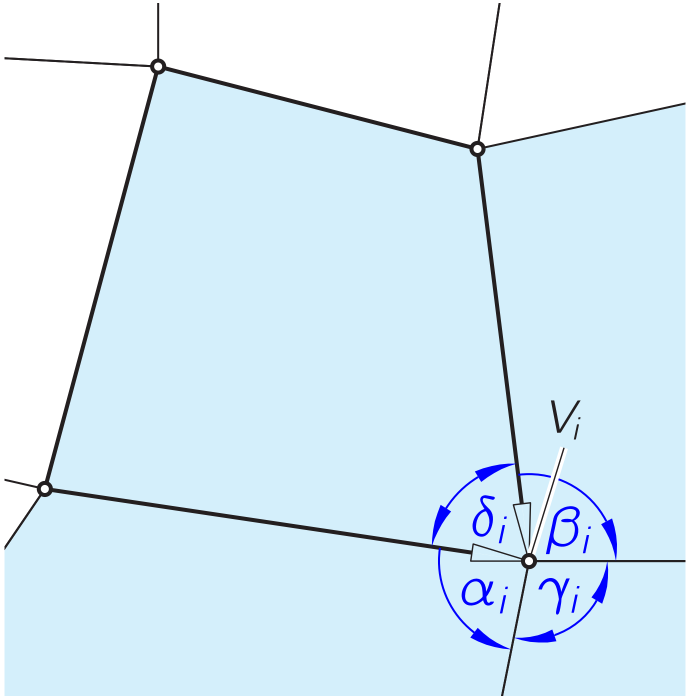


II. Translational type:

There is a translation $V_1 \mapsto V_4$ and $V_2 \mapsto V_3$ mapping the three faces on the right hand side onto the triple on the left hand side.



2. Flexible quad-meshes



III: Isogonal type (Kokotsakis 1932):

A Kokotsakis mesh is flexible when at each vertex V_i opposite angles are either equal or complementary, i.e.,

$$\alpha_i = \beta_i, \quad \gamma_i = \delta_i \ \text{ or}$$

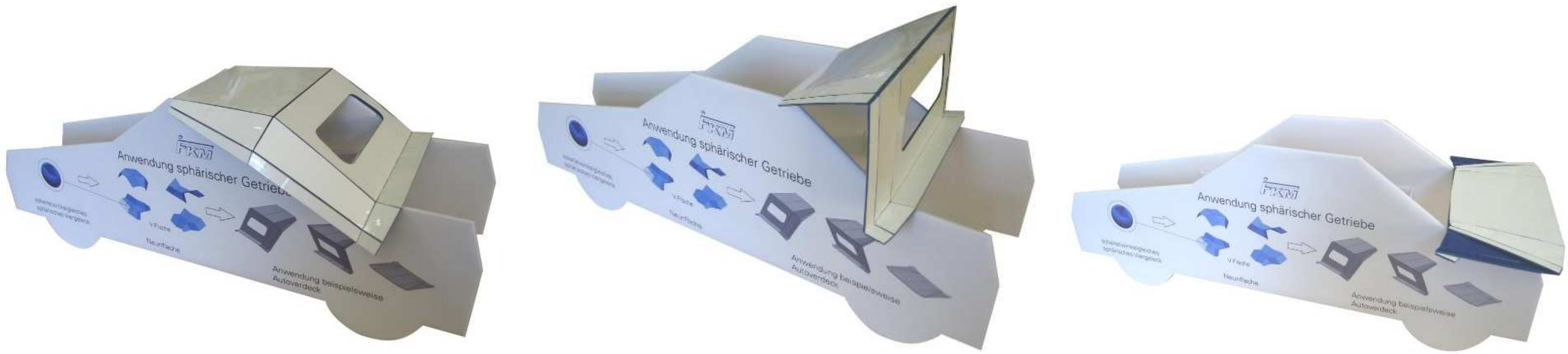
$$\alpha_i = \pi - \beta_i, \quad \gamma_i = \pi - \delta_i \ \text{ and } (n = 4)$$

$$\frac{\sin\alpha_1 \pm \sin\gamma_1}{\sin(\alpha_1 - \gamma_1)} \cdot \frac{\sin\alpha_2 \pm \sin\gamma_2}{\sin(\alpha_2 - \gamma_2)} = \frac{\sin\beta_3 \pm \sin\gamma_3}{\sin(\beta_3 - \gamma_3)} \cdot \frac{\sin\beta_4 \pm \sin\gamma_4}{\sin(\beta_4 - \gamma_4)}$$

A quad mesh where all 3×3 complexes are of this type is continuously flexible and called Voss surface (Kokotsakis, Graf, Sauer)

2. Flexible quad-meshes

These are Voss surfaces:



Nadja Posselt:
Synthese von zwangläufig beweglichen 9-gliedrigen Vierecksflachen
Diploma thesis, TU Dresden 2010

2. Flexible quad-meshes

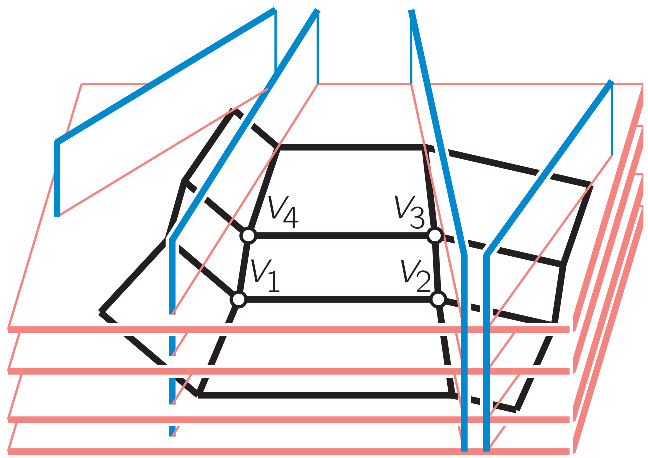
IIIa. Generalized isogonal type:

A. Kokotsakis (1932): At all vertices opposite angles are congruent or complementary.

G. Nawratil (2010): At least at two of the four pyramids opposite angles are congruent.

IV. Orthogonal type (Graf, Sauer 1931):

Here the horizontal folds are located in parallel (say: horizontal) planes, the vertical folds in vertical planes (T-flat).

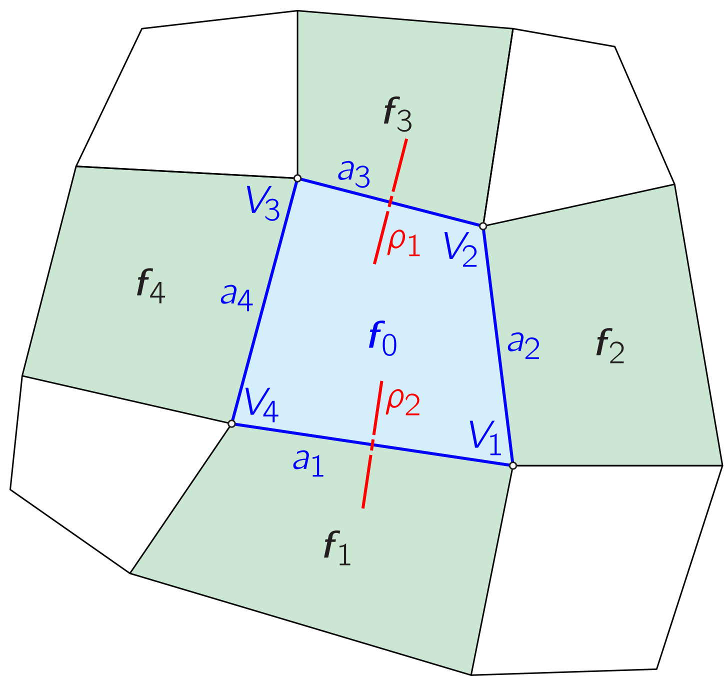


2. Flexible quad-meshes

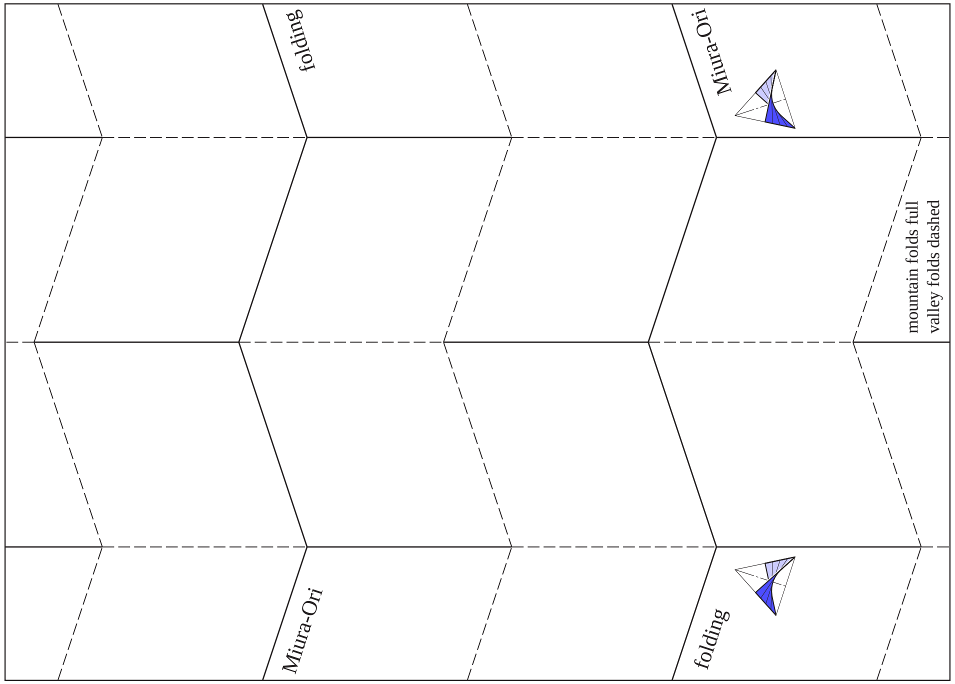
V. Line-symmetric type (H.S. 2009):

A line-reflection maps the pyramid at V_1 onto that of V_4; another one exchanges the pyramids at V_2 and V_3.

This includes Kokotsakis' example of a flexible tessellation.



2. Flexible quad-meshes



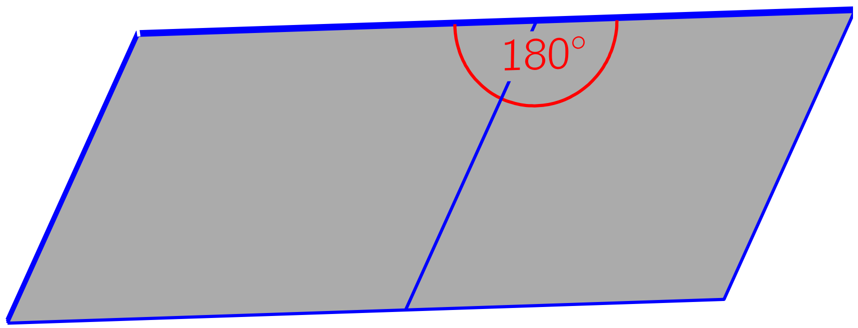
Unfolded miura-ori;
dashs are *valley folds*,
full lines are *mountain folds*

1) Miura-ori is a Japanese folding technique (1970?) named after Prof. Koryo **Miura**, The University of Tokyo (military secret in Russia).

It is used for solar panels because it can be unfolded into its rectangular shape by pulling on one corner only.

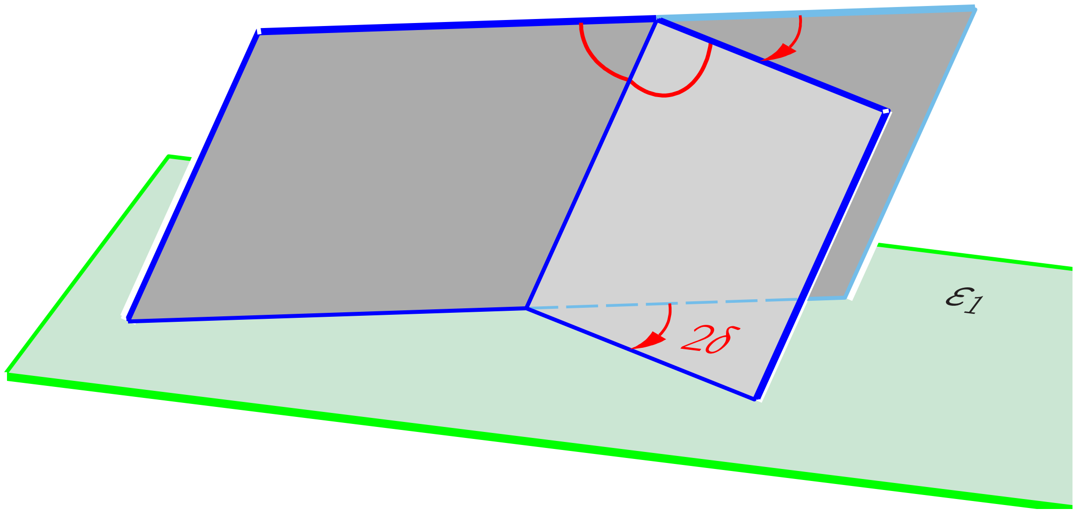
On the other hand it is used as kernel to stiffen sandwich structures.

2. Flexible quad-meshes



we start with two parallelograms sharing one edge . . .

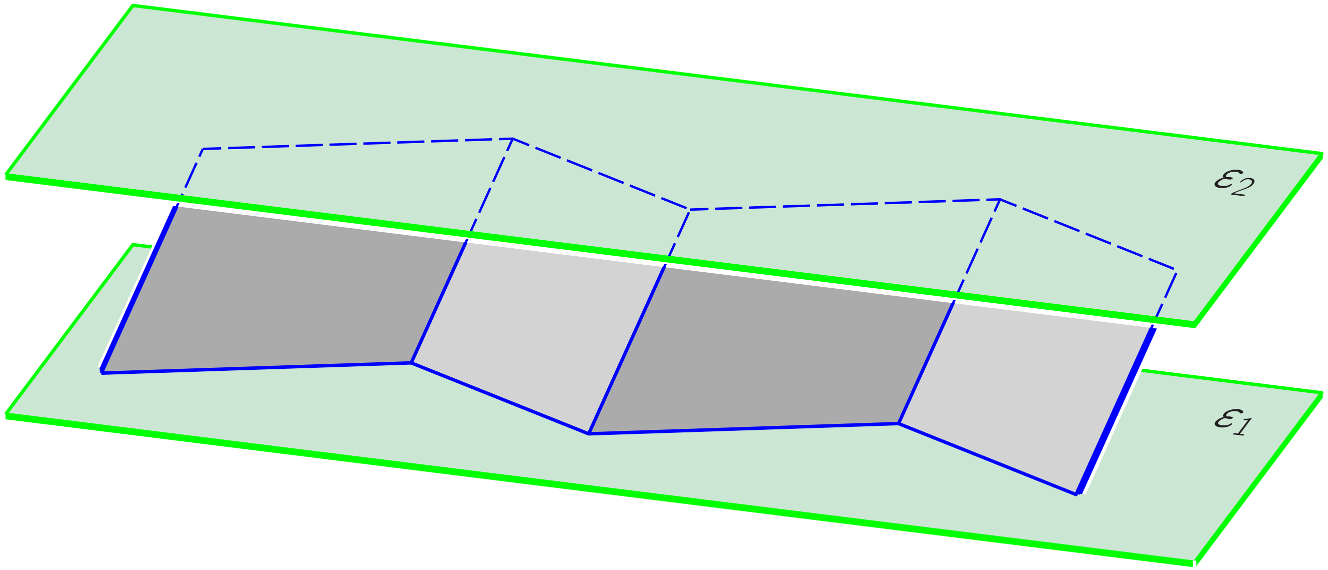
2. Flexible quad-meshes



and rotate the right one against the left one through the angle 2δ.

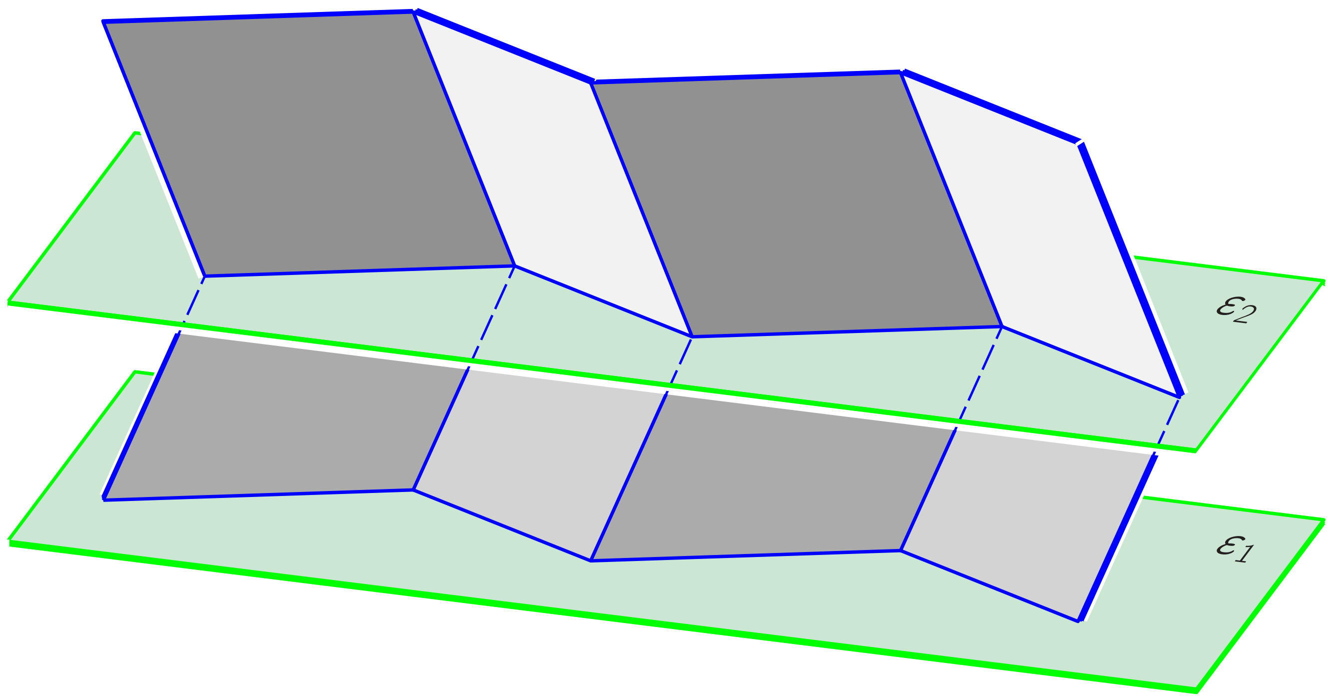
The lower sides span a plane ε_1, the upper sides a plane ε_2 parallel ε_1 .

2. Flexible quad-meshes



By translations we generate a zig-zag strip of parallelograms between the two parallel planes ε_1 and ε_2 .

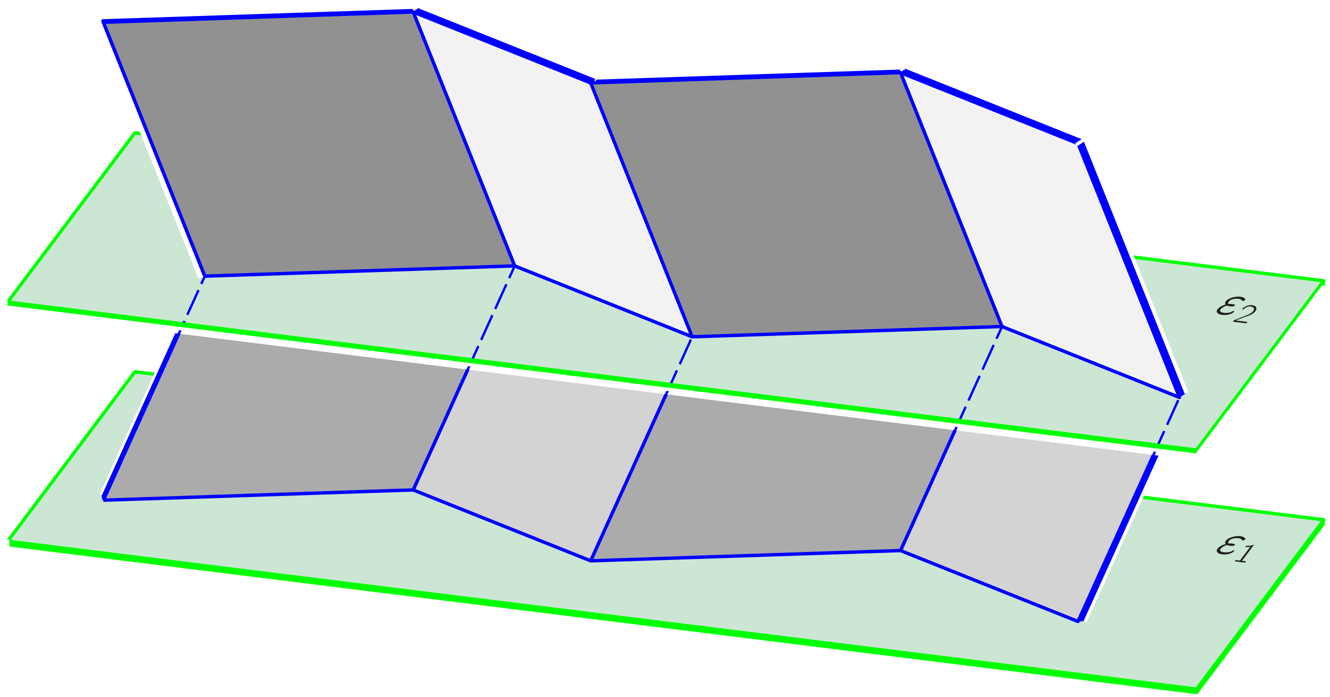
2. Flexible quad-meshes



By reflection in ε_2 we generate a second zig-zag strip of parallelograms sharing the border line in ε_2 with the initial strip

— and we iterate . . .

2. Flexible quad-meshes



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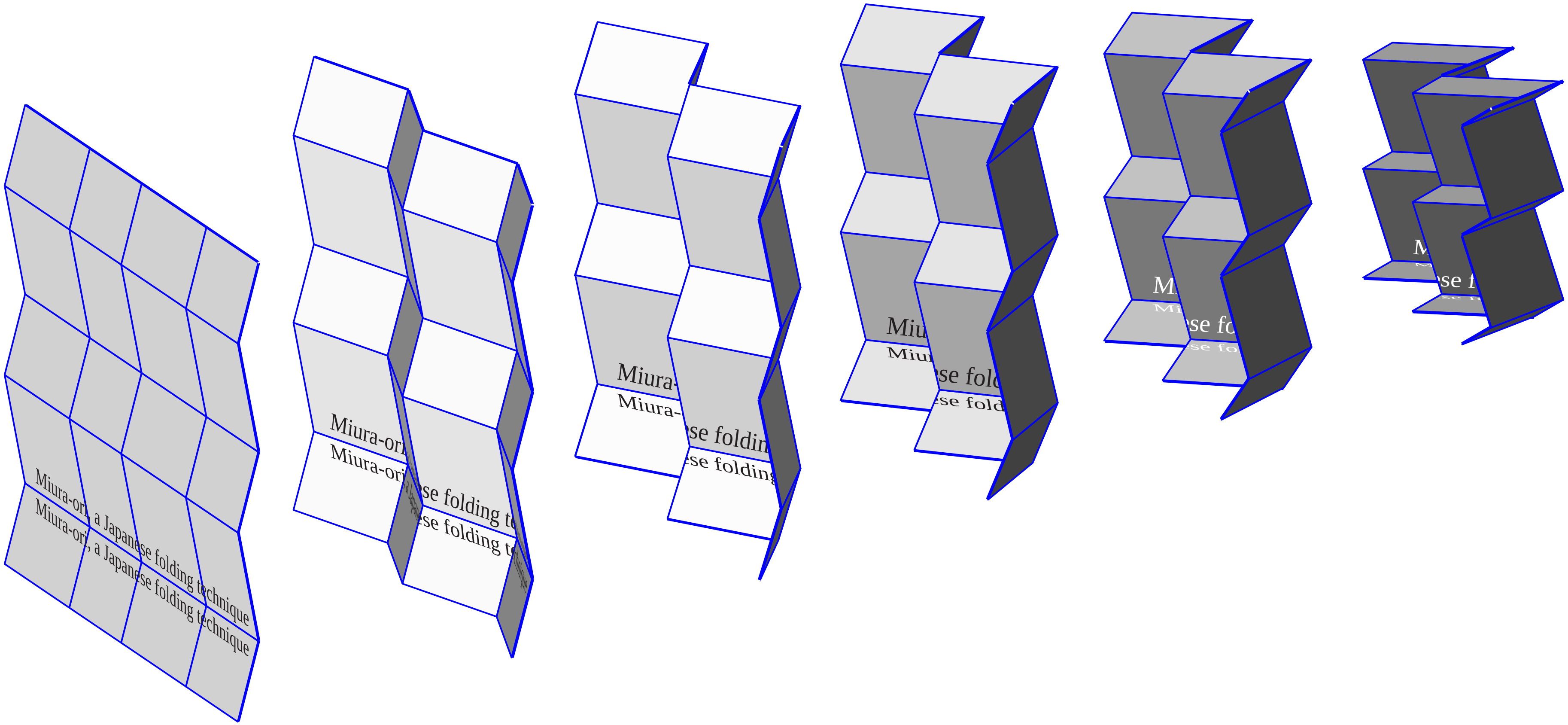

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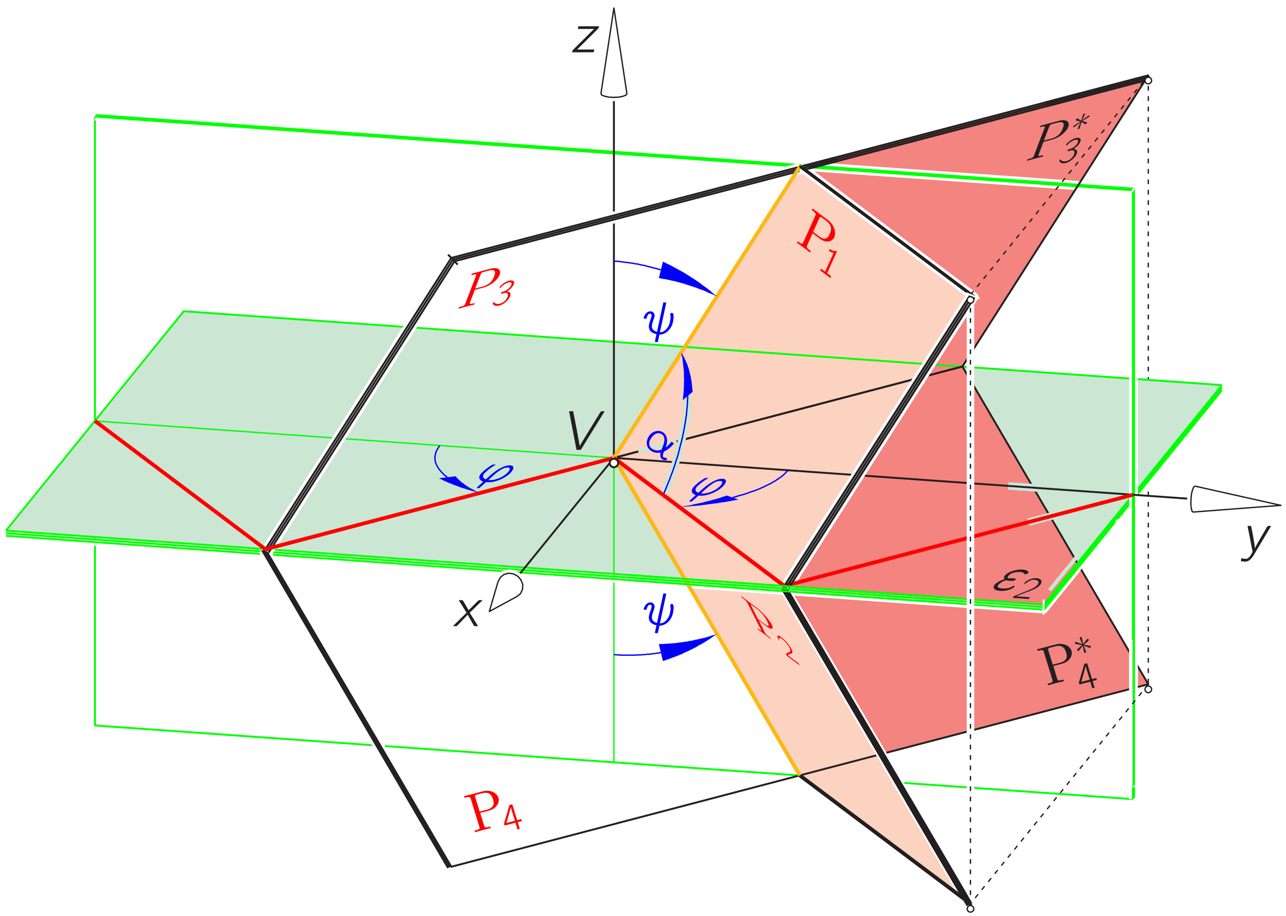
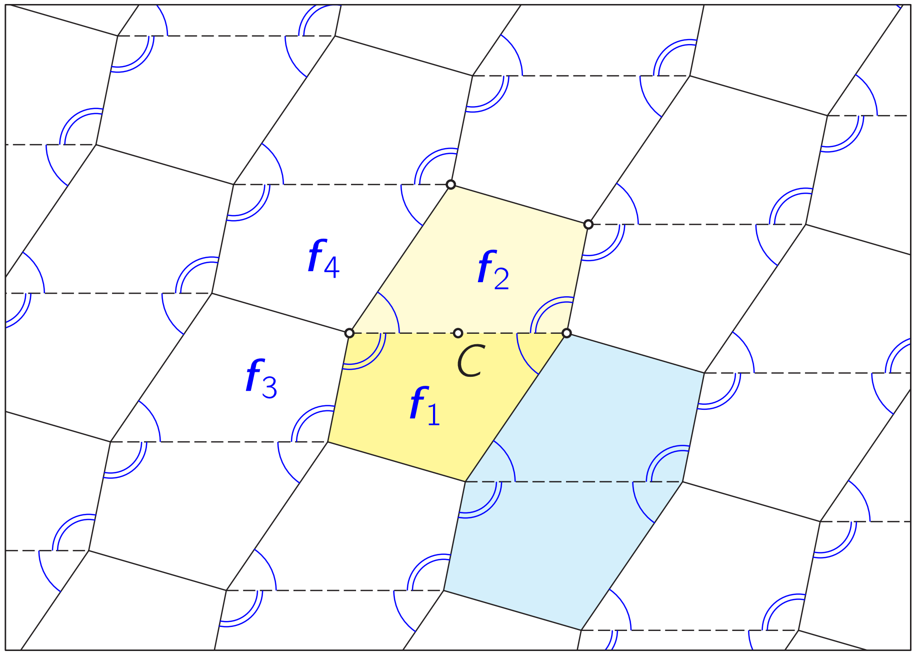


There is a hidden local symmetry at each vertex V:

The parallelograms P_1, P_2 with angle α and the elogations P_3^*, P_3^* of those with angle $180° - \alpha$ form a pyramid symmetric with respect to the fixed planes.

2. Flexible quad-meshes



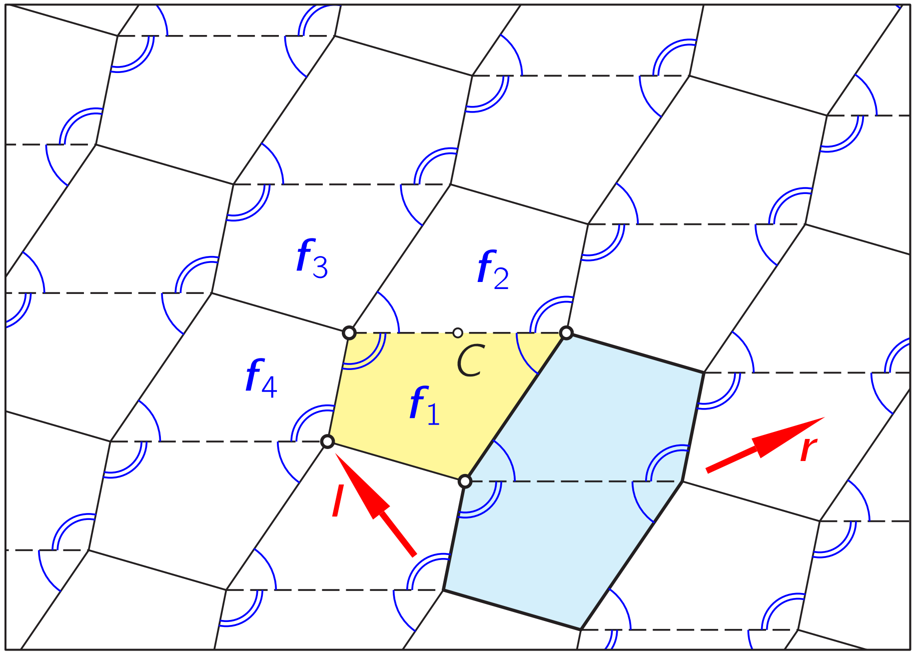
A. Kokotsakis, 1932
Athens

2) Any arbitrary plane quadrangle is a tile for a regular tessellation of the plane. It is obtained from the initial quadrangle

- by iterated 180°-rotations about the midpoints of the sides — or
- by iterated translations of centrally symmetric hexagon.

For a convex f_1 this polyhedral surface is continuously flexible.

2. Flexible quad-meshes



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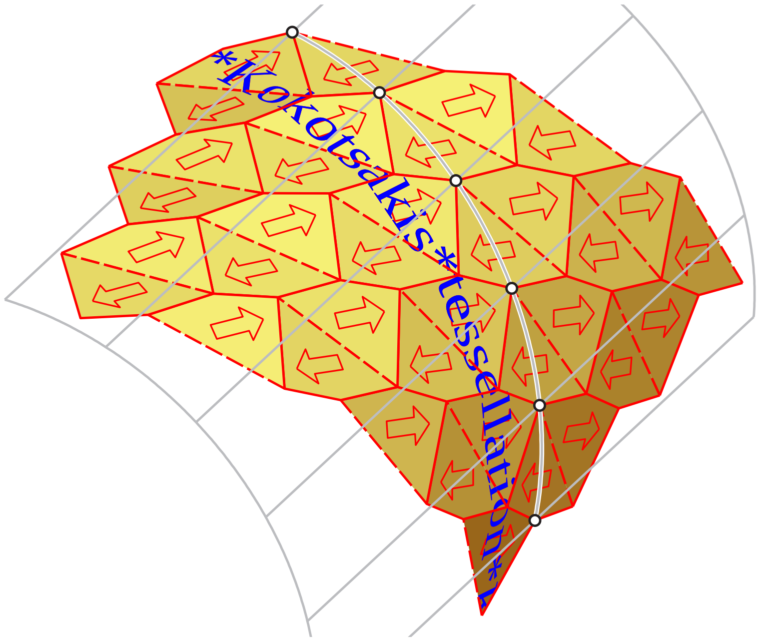
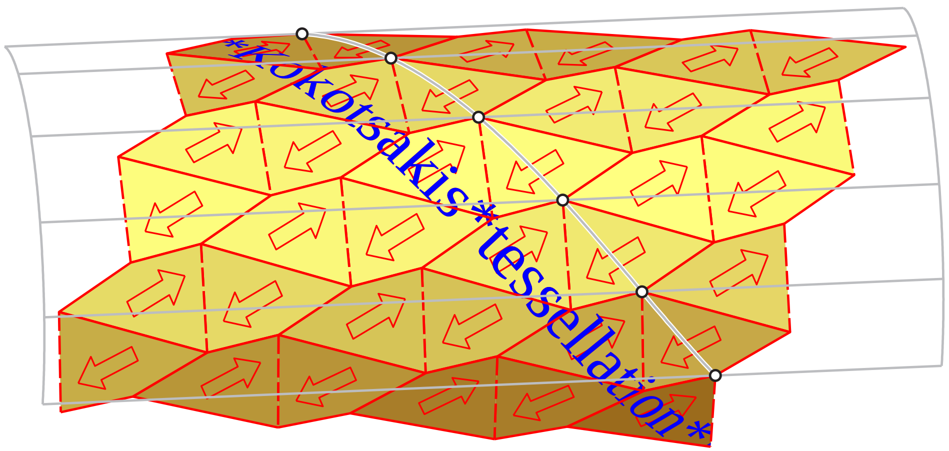
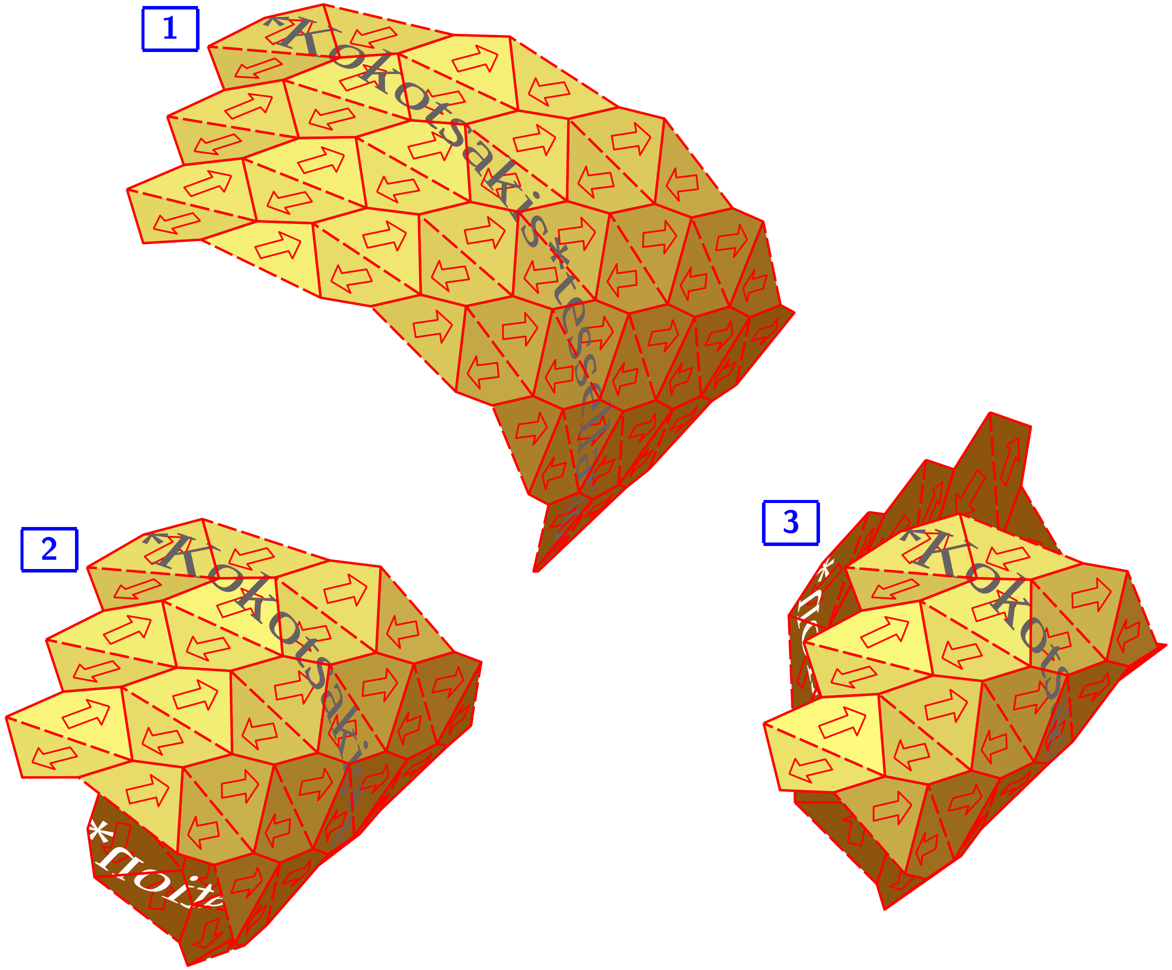




At each flexion all vertices are located on a right circular cylinder.

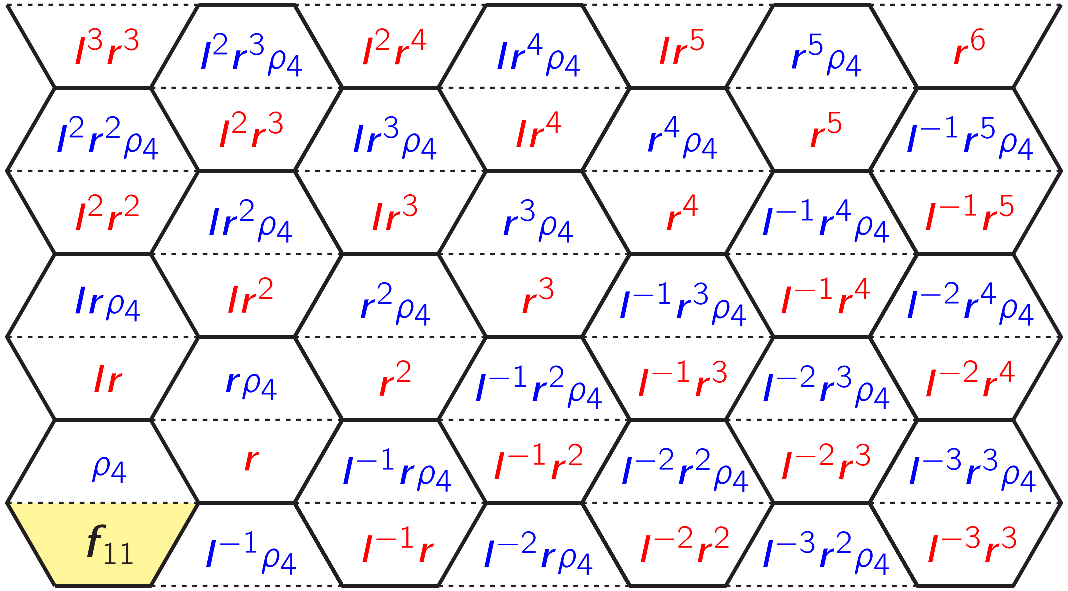
2. Flexible quad-meshes



Different flexions of a 9×6 tessellation mesh (dashes indicate valley folds).

Under which conditions is there a flexion where the right border zig-zag fits exactly to the left border — apart from a vertical shift ?

2. Flexible quad-meshes



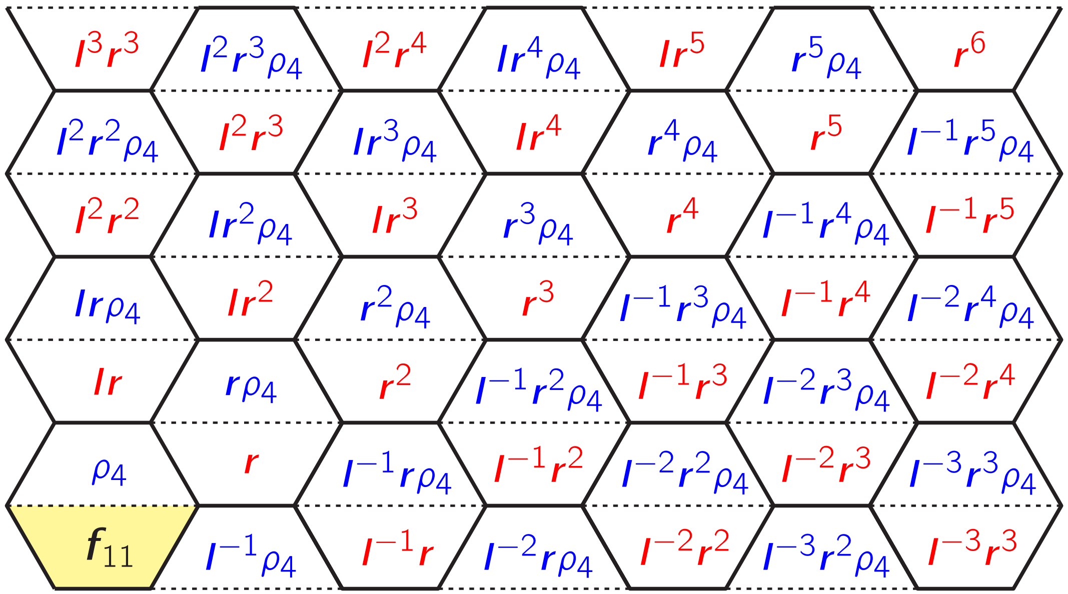
This scheme of a 7×7 tesselation mesh indicates which product of helical motions $\boldsymbol{l}$, $\boldsymbol{r}$ and 180°-rotations ρ_4 maps $\boldsymbol{f}_{11}$ onto $\boldsymbol{f}_{ij}$.

Theorem:

$$\boldsymbol{f}_{ij} = \begin{cases} \boldsymbol{l}^{\frac{i-j}{2}}\boldsymbol{r}^{\frac{i+j}{2}-1}(\boldsymbol{f}_{11}) \\ \text{for } \; i+j \equiv 0 \pmod 2 \\ \boldsymbol{l}^{\frac{i-j-1}{2}}\boldsymbol{r}^{\frac{i+j-3}{2}}\rho_4(\boldsymbol{f}_{11}) \\ \text{for } \; i+j \equiv 1 \pmod 2 \end{cases}$$

$(\, \boldsymbol{r} = \rho_2 \circ \rho_1$ and $\boldsymbol{l} = \rho_4 \circ \rho_1)$

2. Flexible quad-meshes



Coaxial helical motions commute $\Longrightarrow$

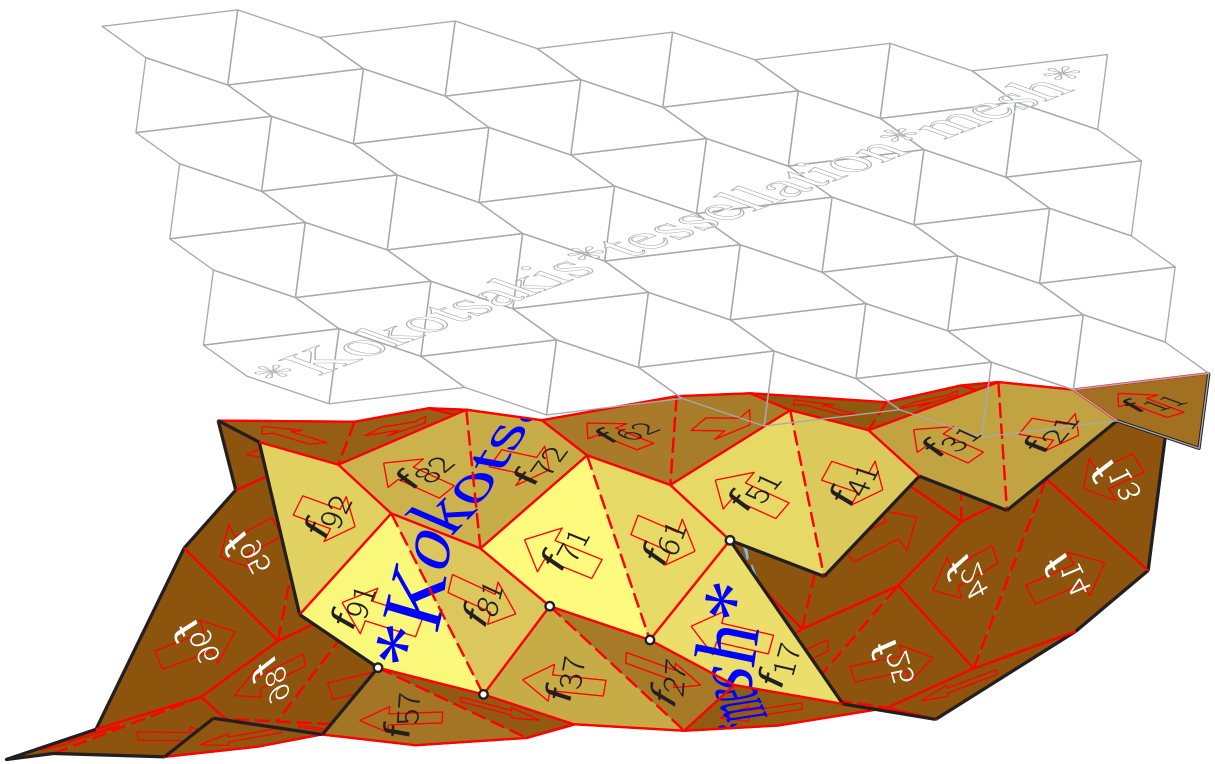
Theorem:
A flexion of a $m \times n$ tesselation mesh closes with a vertical shift of k faces $\Longleftrightarrow$

there exist $a, b \in \mathbb{Z}$ with

$$l^a r^b = d_{2\pi}, \quad k = -a - b.$$

2. Flexible quad-meshes

Closing flexion of a 7×9 tesselation mesh with $l^{-6}r = d_{2\pi}$, $k = 5$.

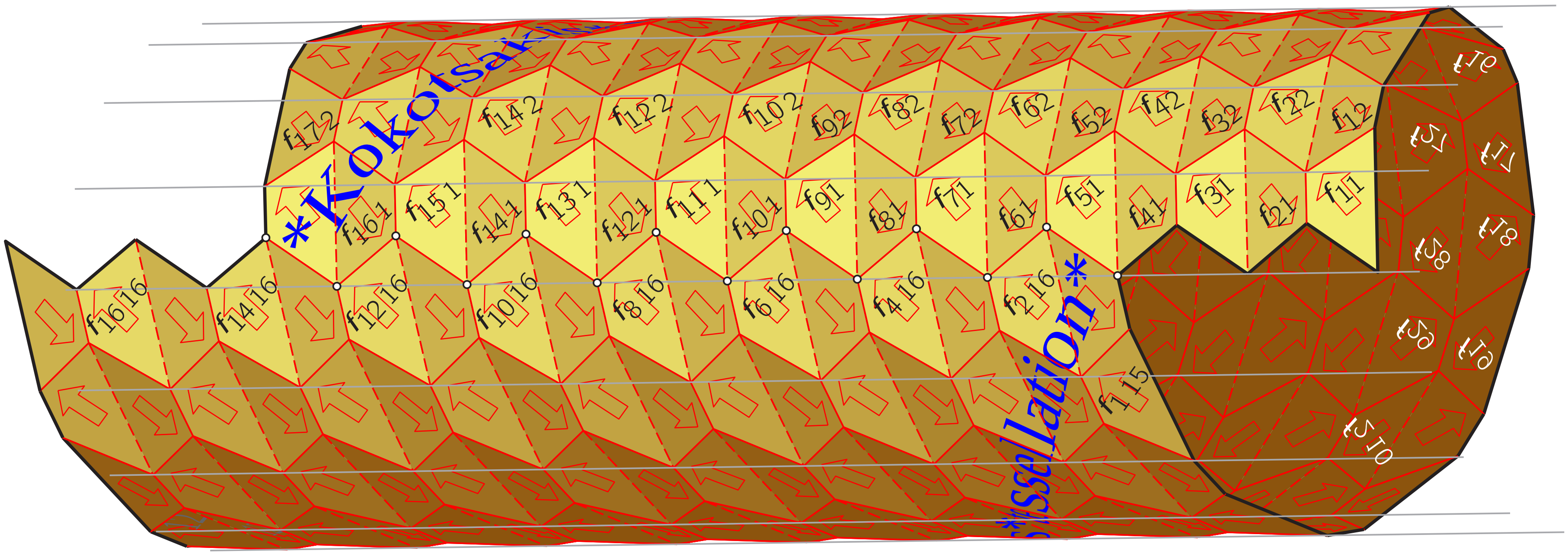


How obtainable ?

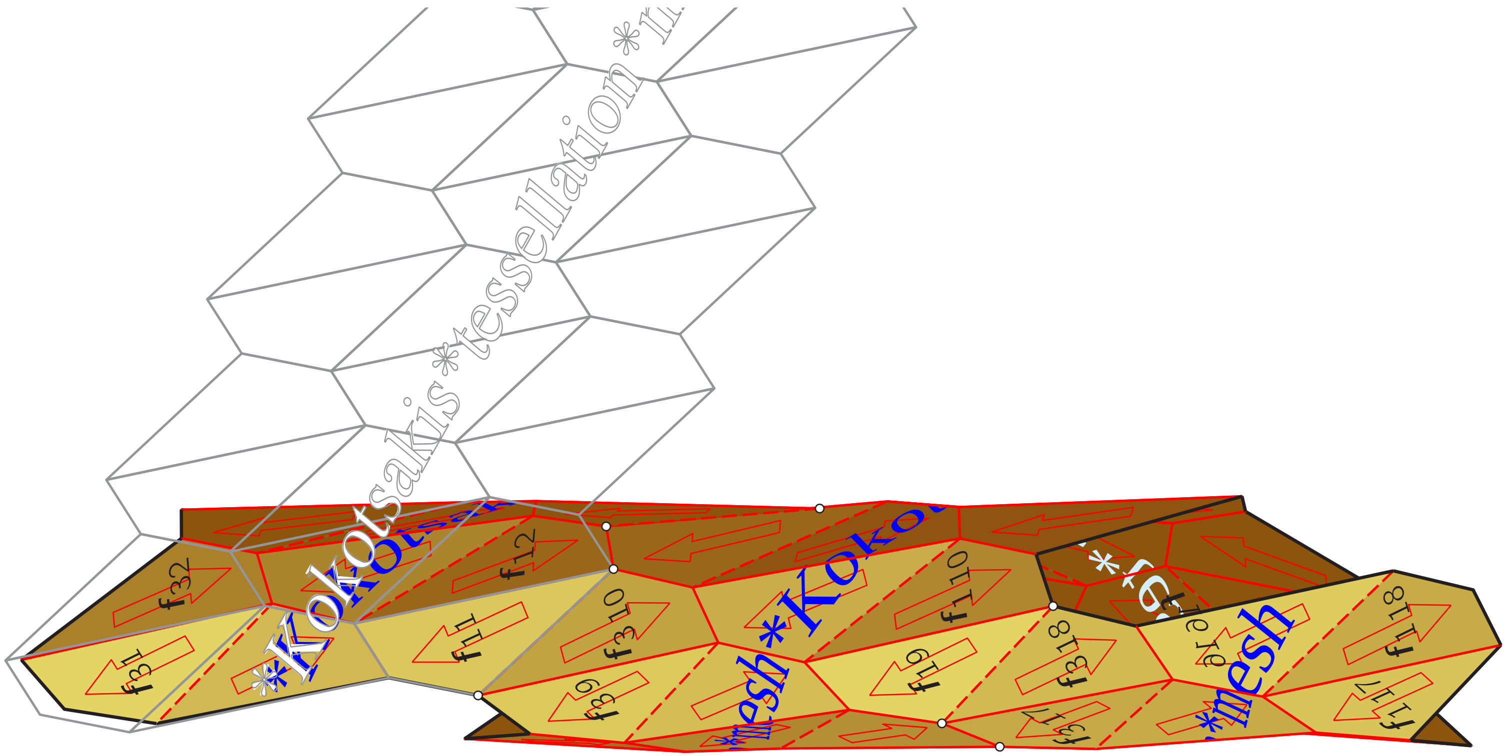
- numerically:
  minimize a distance, or
- start with two coaxial helical motions r, l satisfying $l^a r^b = d_{2\pi}$.

2. Flexible quad-meshes

The analogon of a Schwarz boot with trapezoids; a closing flexion of a 17×16 tesselation mesh with $l^{-10}r^{6} = d_{2\pi}$ and $k = 4$.



2. Flexible quad-meshes



Theorem:
Each closing flexion of a $m \times n$ tesselation mesh is infinitesimally rigid.

3. Curved folding, Example 1

A common way of producing small boxes is to push up appropriate planar cardbord forms Φ_0 with prepared creases. Below the case of creases along circular arcs c_0.

planar version with circular creases

corresponding box with planar creases

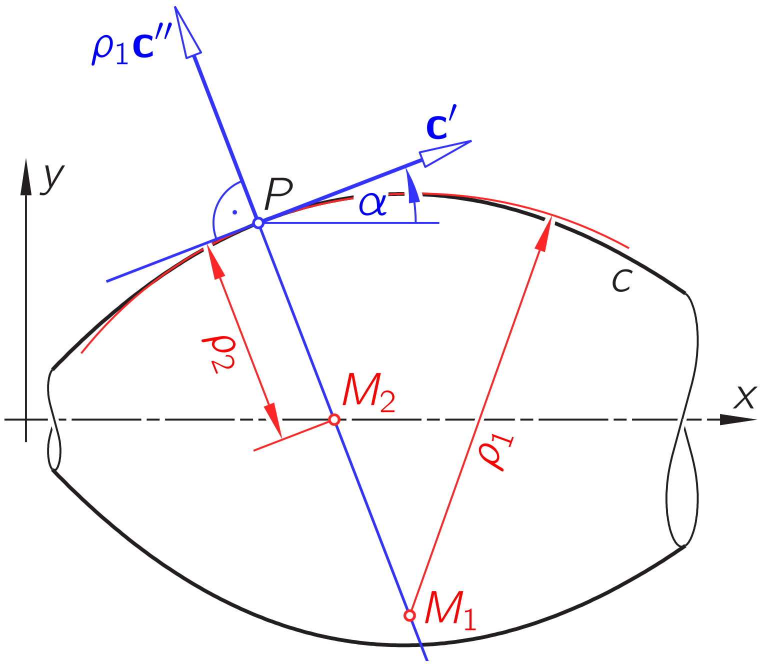
3. Curved folding, Example 1

As proved by **W. Wunderlich** (1958), the spatial creases c are again planar and known as meridians of surfaces of revolution with constant Gaussian curvature.

planar version with circular creases

corresponding box with planar creases

3. Curved folding, Example 1



On surfaces of revolution the meridians and parallel circles are the principal curvature lines. Therefore, the signed principal curvatures are

$$\kappa_1 = -\frac{y''}{\cos\alpha}, \qquad \kappa_2 = \frac{\cos\alpha}{y}.$$

The Gaussian curvature is defined as $K = \kappa_1\kappa_2$. Hence,

$K = \text{const.} \iff$
$y'' + Ky = 0, \ x' = \sqrt{1 - y'^2}.$

provided that $\cos\alpha \neq 0$.

3. Curved folding, Example 1

The general solution of $y'' + Ky = 0$ with constant $K \neq 0$ is

for $K > 0$:

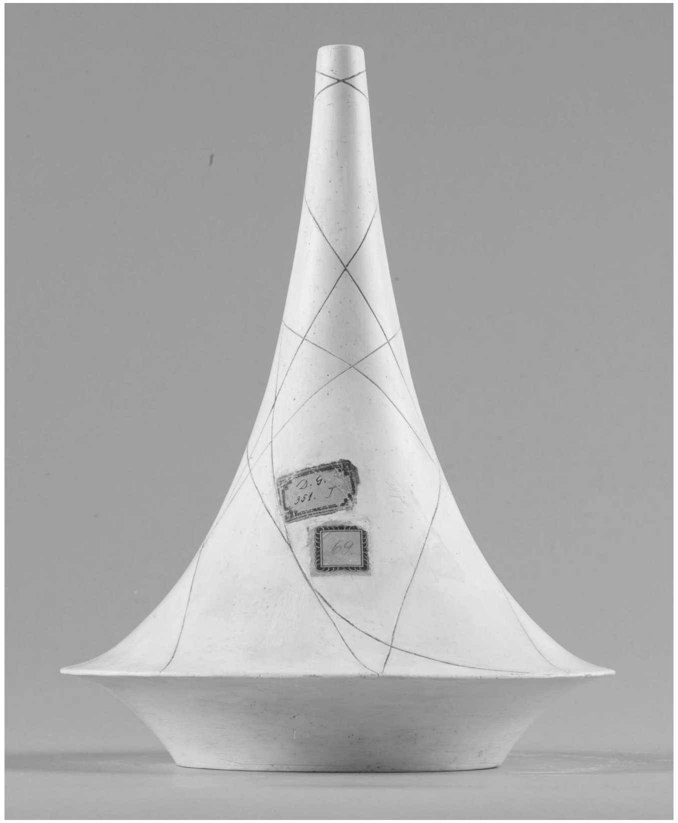
$$y = a \cos s\sqrt{K} + b \sin s\sqrt{K} \, ,$$

for $K < 0$:

$$y = a \cosh s\sqrt{-K} + b \sinh s\sqrt{-K}$$

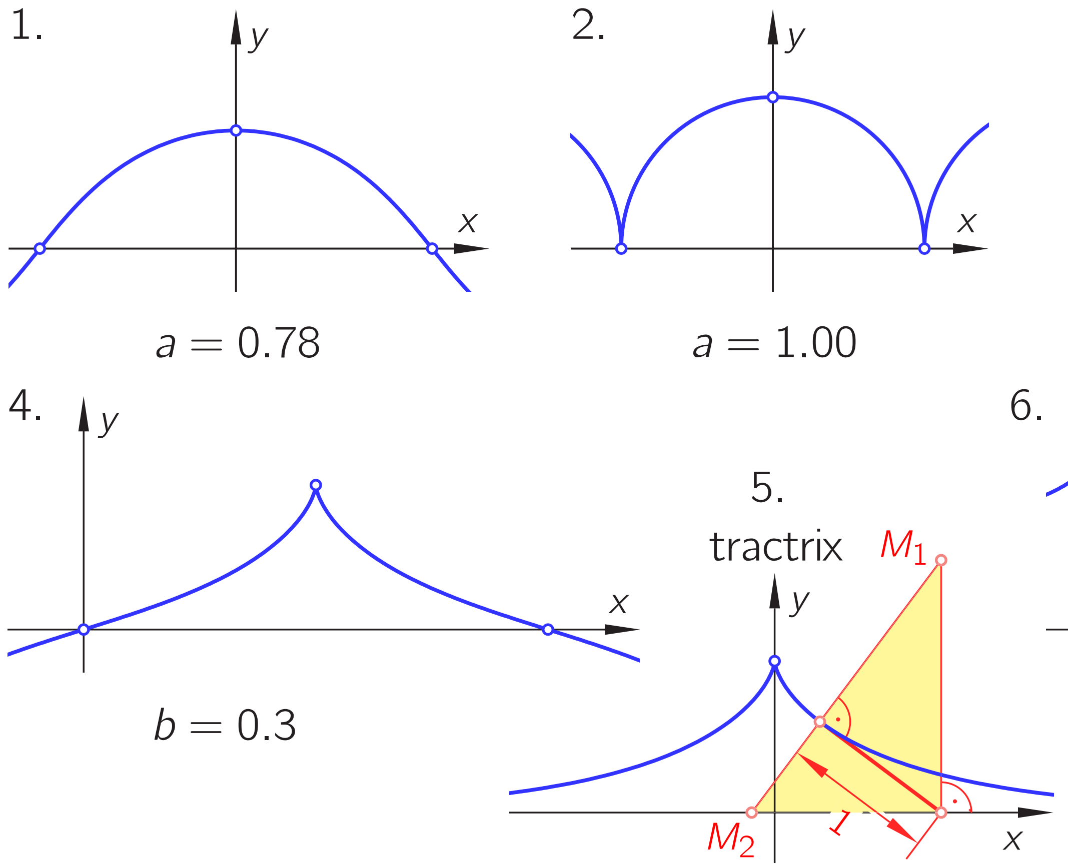
with constants $a, b \in \mathbb{R}$, and $x = \int \sqrt{1 - y'^2} \, ds$.

we can restrict to six cases, up to similarities (Gauß, Minding).



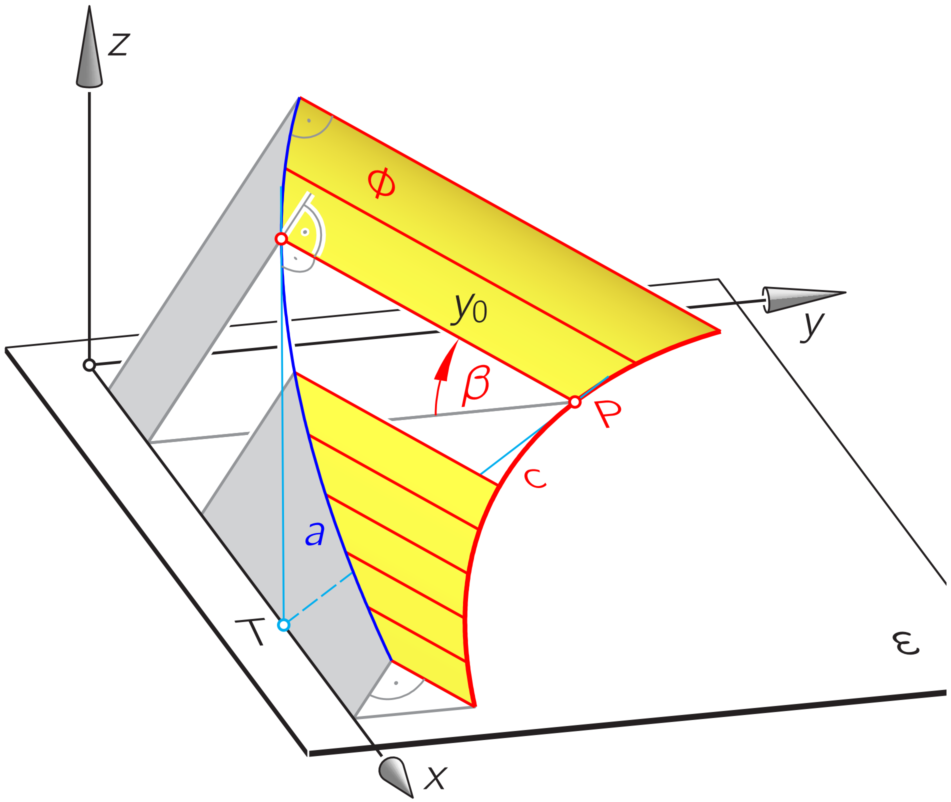
Pseudosphere (tractroid)

3. Curved folding, Example 1



There are six types of meridians to distinguish at the surfaces of revolution with constant Gaussian curvature $K \neq 0$.

3. Curved folding, Example 1



Let c_0 satisfy $y_0'' + K y_0 = 0$ and bound a cylindrical patch Φ_0 with generators orthogonal to the x-axis a_0.

Theorem: If at a cylindrically bent pose Φ of Φ_0 the boundary c lies in a plane ε, then it satisfies the same differential equation as c_0.

Proof: $y_0(s) = y(s)\cos\beta$ with $\beta < \pi/2$ being the (constant) angle of inclination of the cylinder.

The axis of c is the meet of ε and the plane of the orthogonal section a, which is the bent counterpart of the original axis a_0 of c_0.

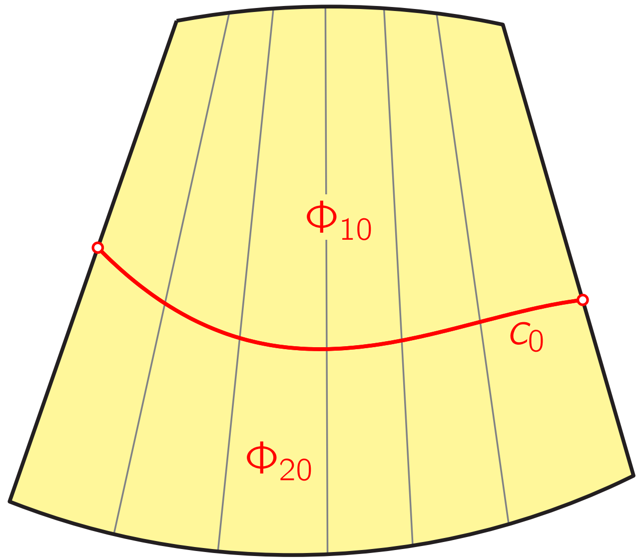
3. Curved folding, Example 1

Theorem:

Let Φ_0 be a planar 'ruled surface' with a transversal curve (crease) c_0, which separates Φ_0 into two patches Φ_{10} and Φ_{20}.

Suppose the generators of the ruling remain straight at the bent pose Φ_1, Φ_2 with a curved edge c between. Then c must be a planar curve.

If all generators of Φ_1 and Φ_2 are extended to infinity, we obtain two torses, which are symmetric with respect to the plane of c.



E.g., take a cone of revolution with a parabolic section c and reflect the part opposite to the apex in the plane of c. In Origami this is called reflection operation.

3. Curved folding, Example 1

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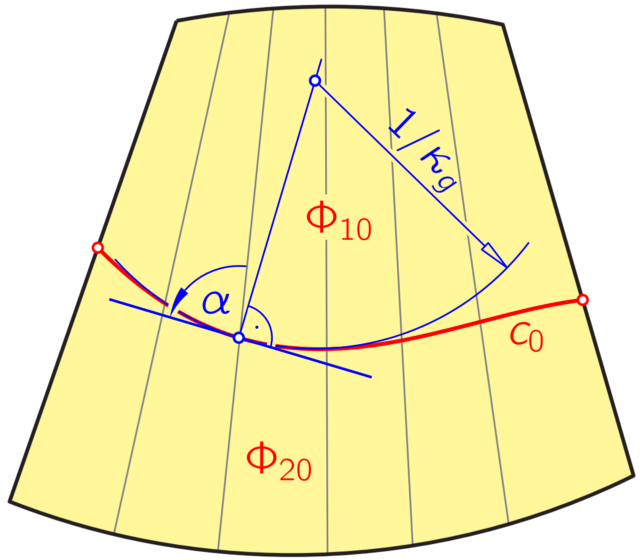

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3. Curved folding, Example 1

Sketch of the *Proof:*

Let $\kappa(s)$ and $\tau(s)$ denote the curvature and torsion of c. In terms of the angle $\gamma_1(s)$ between the osculating plane of c and the tangent plane of the torse Φ_1, the geodesic curvature of c w.r.t. Φ_1 is

$$\kappa_g = \kappa \cos \gamma_1 .$$

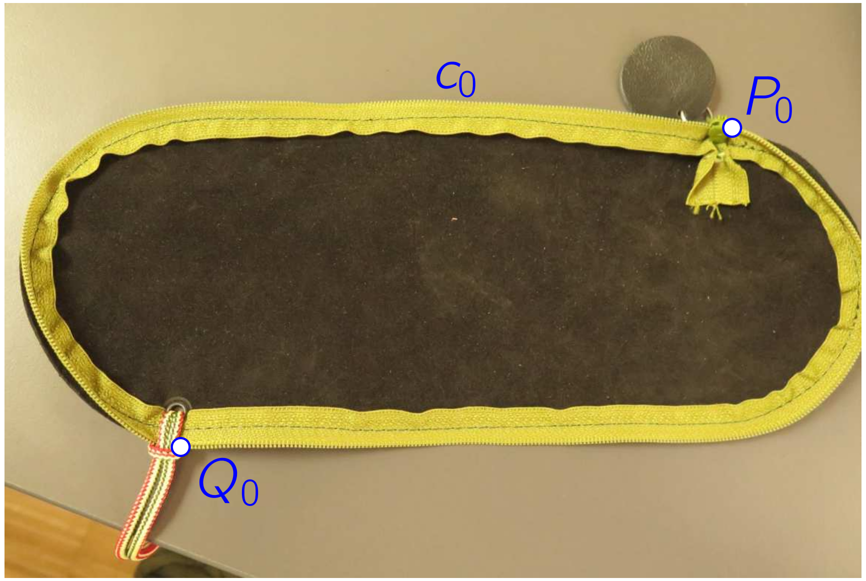
The geodesic curvature κ_g must be the same w.r.t. $\Phi_2 \implies \gamma_2 = -\gamma_1$.

The angle α between the tangent of c and the generator of Φ_1 satisfies

$$\cos \alpha : \sin \alpha = (\tau - \gamma_1') : -\kappa \sin \gamma_1 .$$

The angle α must be the same w.r.t. $\Phi_2 \implies (\tau - \gamma_1') = -(\tau + \gamma_1')$, hence $\tau = 0$.

4. Curved folding, Example 2



Unfolding and corresponding spatial form (photos: **G. Glaeser**)

The spatial form Φ is obtained by gluing together the semicircles with the straight segments. How to model the resulting convex body ?

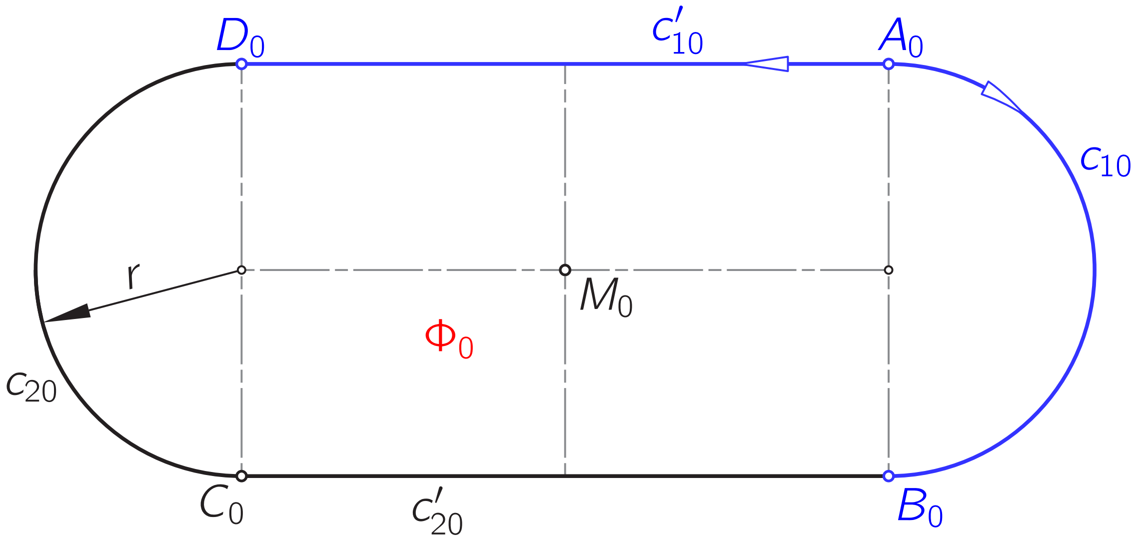
4. Curved folding, Example 2

Unfolding and corresponding spatial form (photos: **G. Glaeser**)

The crucial point is here that the ruling is **unknown.**

M. Kilian, S. Flöry, Z. Chen, N.J. Mitra, A. Sheffer, H. Pottmann: *Curved Folding.* ACM Trans. Graphics **27**/3 (2008), Proc. SIGGRAPH 2008.

4. Curved folding, Example 2



A **physical model** shows:

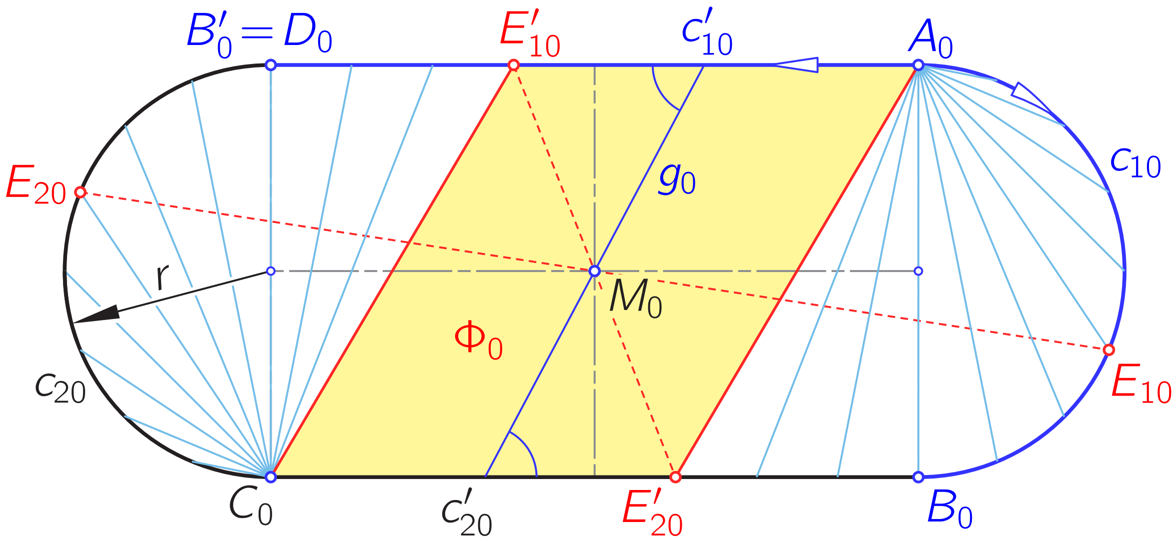
- The spatial body with its developable boundary Φ is convex and uniquely defined.

- The helix-like curve $c = c_1 \cup c_2$ is a proper edge of Φ; the resulting solid is the convex hull of c.

- The semicircular disks are bent to cones with apices A and C. Hence, Φ is a C^1-compound of two cones and a torse between.

- The body has an axis a of symmetry which connects the midpoint M with the remaining transition point $B = D$ on c.

4. Curved folding, Example 2



- At A and C the surface Φ can be approximated by a right cone with apex angle $60°$.

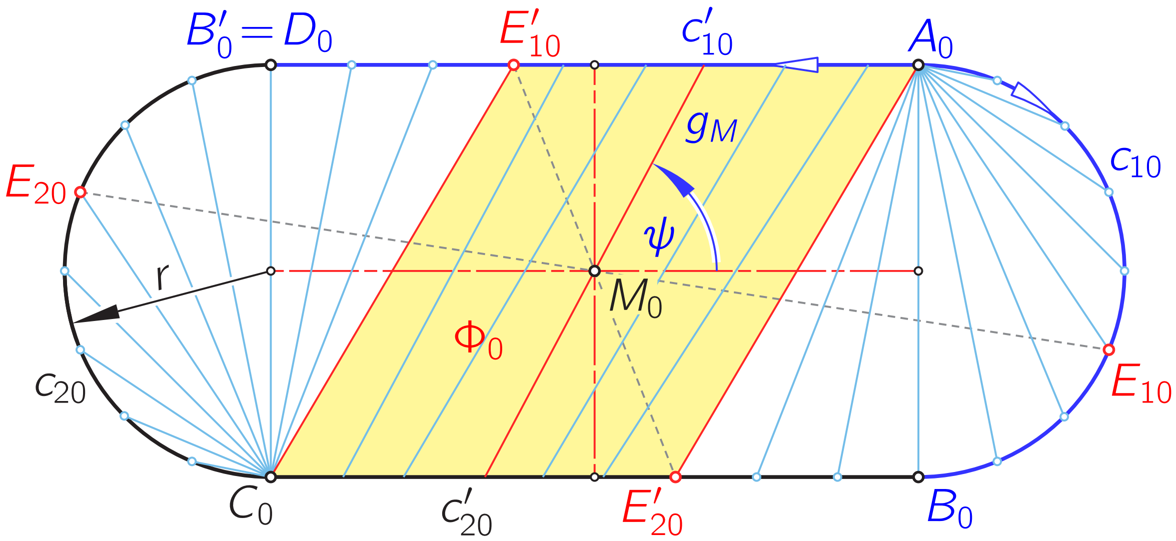
- The tangent t_A to c_1 at A is a generator, the osculating plane of c_1 a tangent plane of this cone; the rectifying plane passes through the cone's axis.

Consequences:

- Because of the straight segments of c_{10}, the developable surface on the left hand side of c_1 belongs to the rectifying torse of c_1.

- When g_0 meets both straight sides of c_0, then g meets c_1 and c_2 at points with parallel tangents $\Longrightarrow$ coinciding tangent indicatrices.

4. Curved folding, Example 2



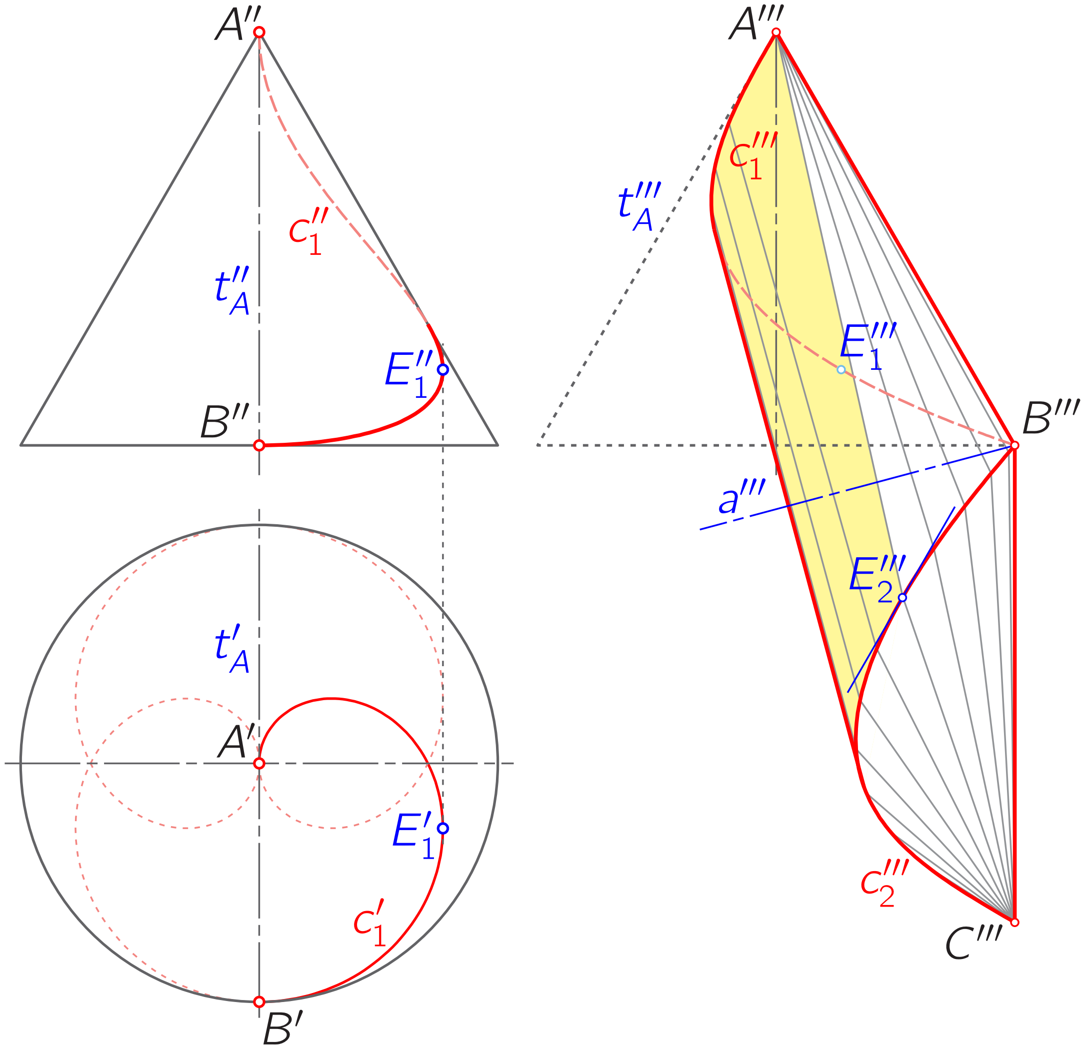
- The tangent at the point $E_2 \in c_2$ of transition between the cone with apex A and the torse must be parallel to t_A.

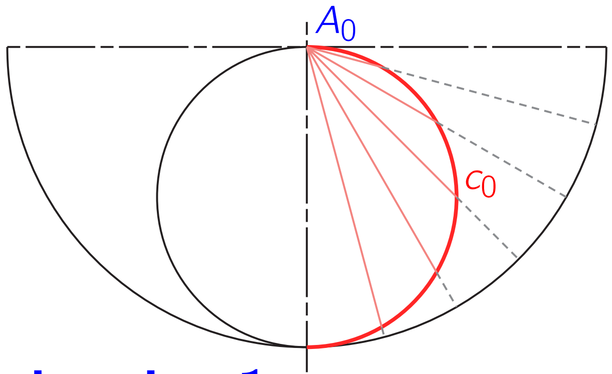
- The tangent at the analogue point $E_1 \in c_1$ is parallel to the final tangent t_C of c_2.

- The subcurves $AE_1 \subset c_1$ and $E_2C \subset c_2$ have conciding tangent indicatrices.

At a **first approximation** the cone with apex A is specified as right cone with apex angle 60°; c_1 is a geodesic circle on this cone.

4. Curved folding, Example 2





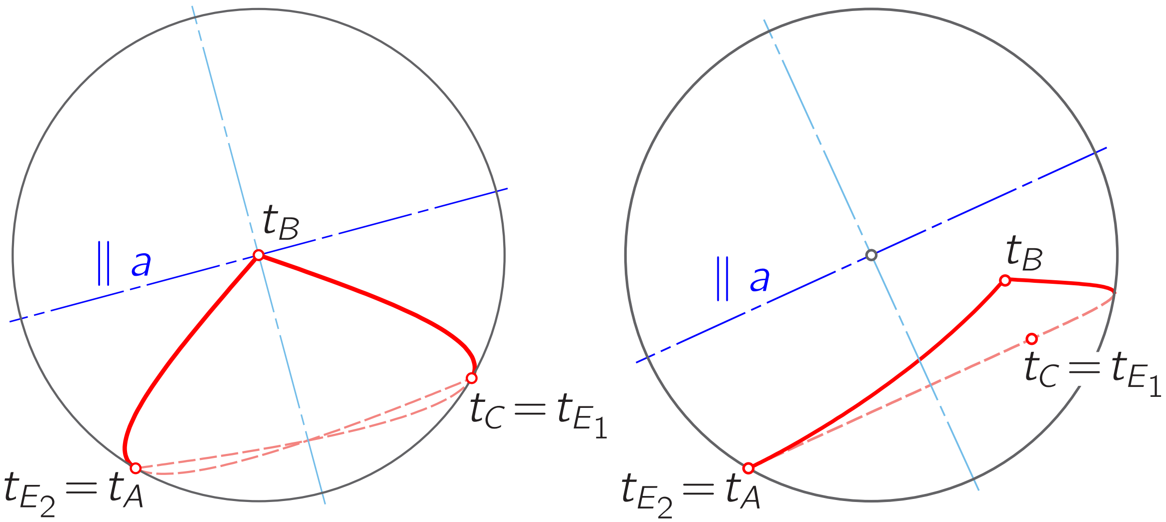
Approximation 1:

c_1 is an algebraic curve.

t_A is parallel to the tangent at $E_2 \in c_2$. Analogously, t_C is parallel to the tangent at $E_1 \in c_1$. This defines the axis a of symmetry.

We notice a contradiction since the osculating plane of c_1 at B is not orthogonal to BC.

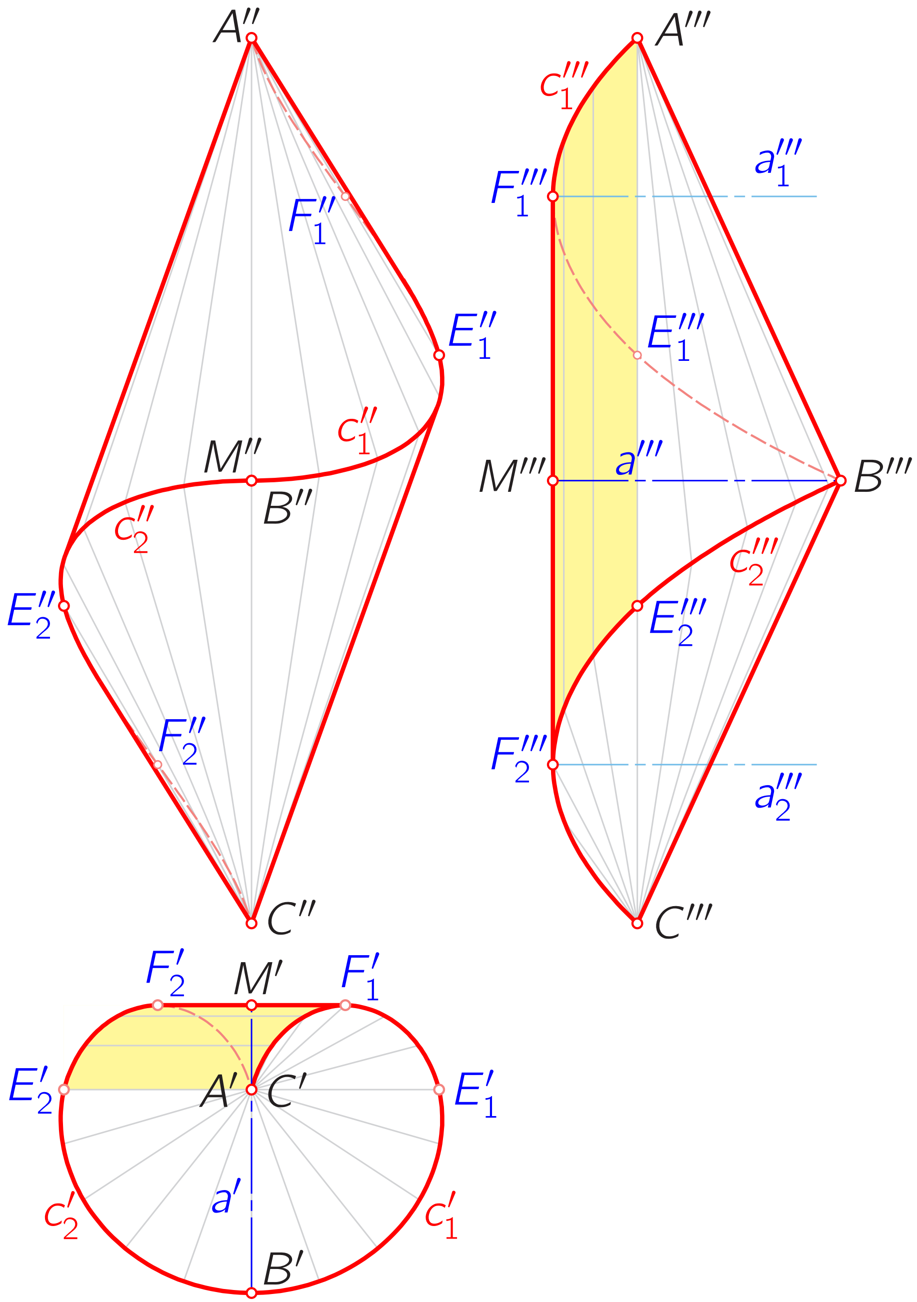
4. Curved folding, Example 2



Left: Tangent indicatrices of c_1 and c_2 for the first approximation; no coinciding subcurves!

Approximation 2 is defined by **alined** side views of the tangent indicatrices (right) $\Longrightarrow$

- the subcurve $AE_1 \subset c_1$ is a curve of constant slope.
- the central torse is a cylinder,
- a translation maps AE_1 onto the subcurve $E_2C \subset c_2$.



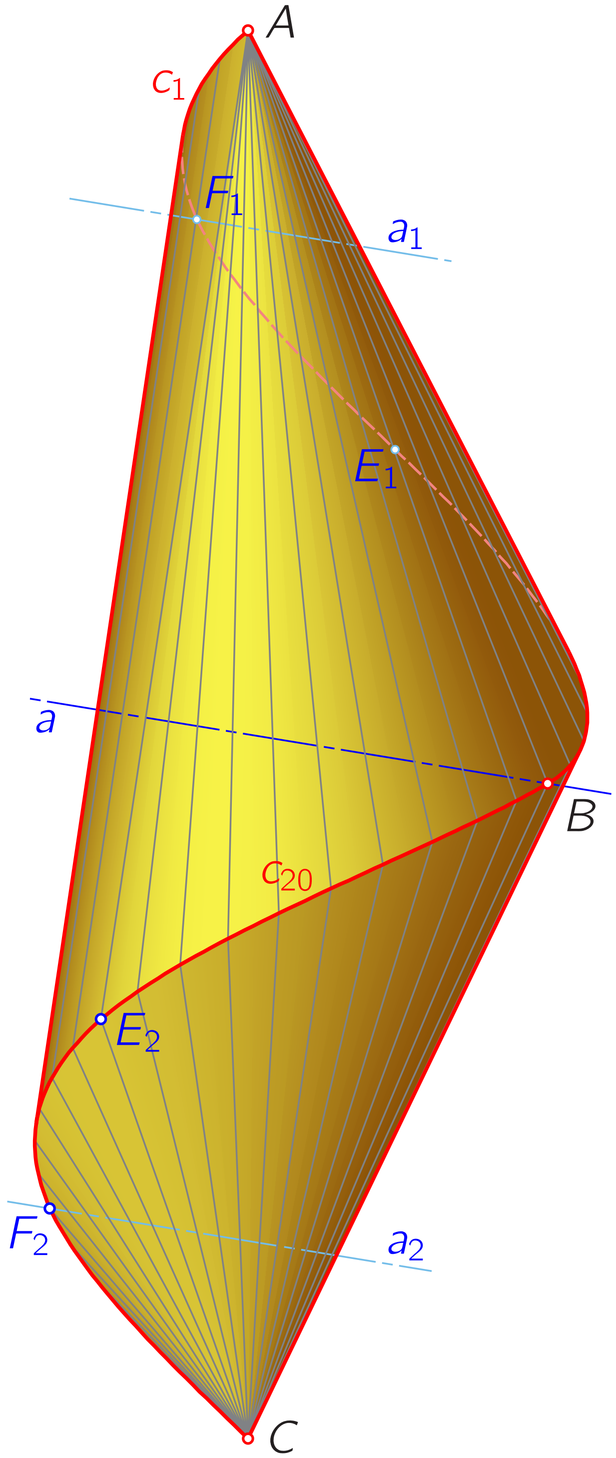
Approximation 2:

The product of the translation $A \mapsto E_2$ and the half-rotation about a maps the subcurve AE_1 onto itself, but in reverse order.

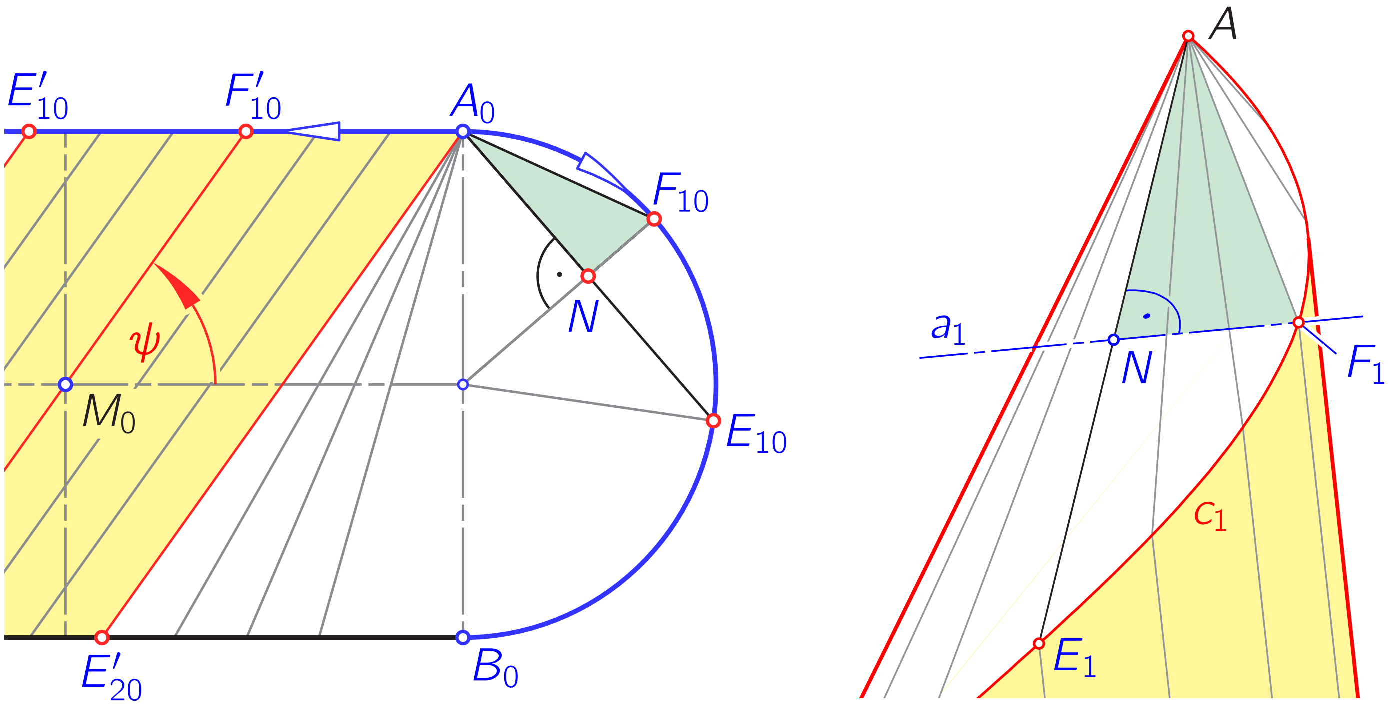
Therefore this portion AE_1 has an axis a_1 of symmetry passing through the midpoint F_1.

Approximation 2 shows an excellent accordance with the physical model.

... but there remains a contradiction.



4. Curved folding, Example 2



Due to the symmetry w.r.t. a_1, the midpoint N of AE_1 lies on a_1. The distances $\overline{A_0F_{10}}$ and $\overline{A_0E_{10}}$ are preserved, the triangle ANF_1 is congruent to its counterpart $A_0N_0F_{10}$ in the unfolding. But NF_1 is not (exactly) orthogonal to the tangent of c_1 at F_1.

Thank you for your attention!

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